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Design of R-C-C-R spatial mechanism

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DESIGN OF R-C-C-R SPATIAL MECHANISM

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ABSTRACT

An analysis to design a particular R-C-C-R spatial mechanism was developed using tensor notations and operations. This specific R-C-C-R spatial mechanism can be used to couple two shafts intersected at a skew angle with constant rotating speed ratio. This skew angle, or bending angle of the two shafts, may vary from 30 up to 180 degrees. The governing equations of relative positions of the mechanism were formulated as a function of two variables: the bending angle of the shafts and the rotating angle of the input link. To exemplify the various design conditions, the design parameter versus the bending angle was tabulated. To ensure dynamic stability of the system, a symmetrical design was considered. A modified R-C-C-R mechanism was also suggested to couple shafts of varying intersection angles. This mechanism may replace the bevel gear, Hook's joints, or other kinematic pairs for the purpose of indirect transmission.

In order to exemplify the convenience of tensors in kinematic analysis, the investigation of the instantaneous relationship between input and output links of eight feasible mechanisms (in appendix) were included. Kinematic solutions were also discussed and computerized for a R-C-R-C-R spatial mechanism.

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CHAPTER I

INTRODUCTION

In recent years, research on three-dimensional mechanisms has been quite intensive. It is because its extra dimension gives kinematicians more design parameters than the traditional two-dimensional mechanisms. Due to the rapid development of high speed digital computers and various elegant mathematical methods, achievement of spatial mechanisms has become considerable.

In Germany, analysis and synthesis techniques were performed by Altman [1]*, Beyer [2], Hain [3], Keler [4], and others, based on graphical and analytical geometry of dual quantities. In U S S R, the tensor analysis was used by S. G. Kislitsyn [5] in the theory of spatial mechanism. While in this country, Denavit and Hartenberg [6] adopted the matrix calculus; Chace [7], Beyer [2] and Harrisberger [8] used the vector technique; Yang and Freudenstein [9] chose matrices with dual-number quaternions and Ho [10] developed the tensor method.

It seems very convenient if a spatial mechanism is pursued by tensor methods. In a spatial mechanism, almost every quantity involved can be expressed by tensor notation; such as scalar quantity is a tensor of zeroth order, velocity and acceleration are the tensor of first order, moment of inertia is the tensor of second order,

*Numbers in parenthesis refer to the Bibliography at the end of the thesis.

and other complicated quantities can be represented as the tensors of higher order. Most spatial mechanism problems can be easily solved and have closed-form solutions using the tensor operations with various geometrical constraints imposed upon the mechanism. In order to show the convenience of the technique, the author has also illustrated the way to find the instantaneous relationship between input and output links of eight feasible mechanisms. They are listed in the Appendix. The kinematic properties are solved by differentiating the instantaneous position relationship equations. The torque transmittion is evaluated through energy method [11]. They are demonstrated in a R-C-R-C-R spatial mechanism.

Graphical and related methods have been important because they avoid detail computation and provide a visual perspective. With the advent of the digital computers, the computational advantage of graphical methods becomes less significant. And as we all know, it has always been difficult and tedious to apply graphical methods to three-dimensional analysis.

Analysis and synthesis by conventional complex mathematics has been very successful for two-dimensional problems. Extension to three-dimensional analysis was suggested by Raven [12] but nevertheless conventional complex mathematics has remained a two-dimensional tool.

Matrices methods have been developed and applied by Hartenberg, Denavit, and Uicker [13]. A computer program, based on this method, will obtain position, motion and force solutions for the complete motion cycle of three-dimensional mechanism connected by lower pairs in any single loop. The method has been extended to general spatial mechanisms. However, the iteration technique required for the position solutions is always subjected to the truncational errors of the computations. And interpretation of the matrix equations is also difficult. Nevertheless, it is a beautifully formulated approach and it affords the available numerical solution to an important category of mechanisms.

Dual numbers, quaternions, and vectors have been applied to this kind of problem. These approaches have advantages in the representation of spatial problems, but any advantages they may have have not been made clear especially for complicated linkages not possessing geometric symmtry.

The Instantaneous Screw Axis (ISA) theory adapted by Skreiner [14] is a newly developed method which has a good approach to the problem. However, the positions of the links relative to the ground link are not easily determined through this method. It, therefore, is discarded in favor of the more direct tensor methods introduced here.

The goal of machine design is to find a simple but effective mechanism for transmitting a desired motion. Spatial mechanisms have long been avoided by designers because of their complexities. The construction of a spatial mechanism is amazingly versatile. Sometimes, an ingenious arrangement will be able to simplify a complex mechanism and accomplish the desirable motion.

In this paper, the author has adapted a very simply R-C-C-R spatial four-bar mechanism with two revolute pairs and two cylindrical pairs. This specific mechanism can be used to couple two intersected shafts, with one-to-one speed ratio. Practically, the angle of intersection or bending angle may range from 30 to 180 degrees.

Actually, the R-C-C-R spatial four-bar mechanism may be visualized as a combination of two similar R-C-R-C-R* spatial five-bar mechanisms without the portions of "R-C-R" connected them oppositely. And it can also be modified to be another <u>R-C</u>-R-<u>C-R</u>* spatial mechanism. This modified mechanism thus may couple two shafts that intersect with variable bending angles.

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*<u>R-C</u>-R-<u>C-R</u> spatial mechanism is a modified R-C-C-R spatial mechanism symmetrical to the middle pivot joint. The underlines denote symmetry, while R-C-R-C-R spatial mechanism is a completely different mechanism even though the same name is used.

It is hoped that this paper will demonstrate the convenience of adapting the tensor technique in spatial mechanisms and serve as a basis for discussion and further generation of ideas.

CHAPTER II

LITERATURE REVIEW

DEFINITIONS AND BACKGROUND MATERIAL

Before advancing to the rigorous analysis of spatial mechanisms, it would be well to review a few important definitions and concepts to form a general basis on which to proceed.

A. KINEMATIC CONCEPT

1. Mechanism

A mechanism is a device to transform one motion into another. If the transmission also includes substantial forces, the device is a machine. If forces are associated with the conversion of the energy of fluids (as steam, water or gas) to shaft power, then the aggregate may be called an engine. In any event, it is recognized that the elements or parts of the device must be resistant to deformation i.e., they must be approximately rigid bodies. Thus we may then say that a mechanism is an assemblage of rigid or resistant bodies connected in some way for the purpose of transferring certain desired motion.

Two general groups of mechanism exist, since the motion may go from uniform to either uniform or nonuniform, hence they may be divided as uniform motion converters and nonuniform motion converters. Circular gears, chains, belts, and the like, comprise most of the former. The later are made with noncircular gears, cams, ratchets and some linkages, both planar and spatial. An axial cam rotary engine is an example of this.

2. Motion

Ordinarily, there are two kinds of motion, one is translation and the other is rotation. Both of them include at least three distinctly different yet related characteristies, i.e., displacement, velocity, and acceleration, requiring a certain amount of time for their completion.

Whenever a motion is described, the reference frame with which the motion is related must be defined first. In mechanisms, it is the very fact that motion exists implies reference frames of some sort on the moving parts or a certain specified fixed coordinate. If the reference frame of one machine part moves with respect to the reference frame of another, we speak of relative motion. When the reference frame of one part is fixed with respect to the ground and the motions of the other parts are referred to it, then these particular relative motions may be termed absolute motions. Absolute motion is thus a special case of relative motion, in which the reference frame is fixed.

3. Link

In kinematics, the members of which a mechanism is made are called links. In order to have definite motion, all concepts of deformation due to load and all stress-strain relationships are neglected. The only purpose of the links is to hold a specific relationship between the several pairs or joints. The particular physical configuration, the shape, the size, the weight, the material, are all only incidental in the study of how a mechanism will move. A link is then a rigid body containing the elements of at least two pairs.

4. Kinematic pair

A mechanism has been defined as a number of rigid bodies so connected that each moves with respect to another. These connections or joints between links are called pairs. The clue to the nature of a mechanism lies in how the pairs are connected and what kind of relative motion the connection allows. In kinematics, a pair is a joint between two members permitting a particular kind of motion. We should note that the term connection, in engineering fields different form kinematics, may mean an immovable connection, as a structural joint or splice, shrink fit, and the like.

The pair variables are the variable parameters which must be measured to describe the relative motion permitted. For instance, the angle of rotation of a hinge is its only pair variable. A particular pair may have up to five pair variables, depending on the number of degrees of freedom involved.

Kinematic pairs, although they are often difficult to recognize by their physical appearance, may be categorized according to their characteristic motions into three main groups: lower pairs, higher pairs, and wrapping connectors. The lower pairs are recognized by the particular kind of relative motion permitted the connected links, that is defined by an clearily associated pair variable, or by a simple grouping of functionally indirectly-related pair variables. They are six in number: the revolute pair (R), which permits rotational motion about an axis and is defined by a single variable angle θ; the prismatic pair (P), which permits translation motion in one direction and is defined by a single displacement variable S; the screw pair (H), which permits helical motion and is defined by either the rotation angle θ or translation displacement S, related through $\theta/2\pi = S/h$, where h is the lead of the screw (advance per revolution); the cylindrical pair (C), which permits a translation motion parallel to an axis and also a rotation motion about the same axis independently; the spheric pair (S), which permits a globular rotation about a point; and the planar pair (F), which permits planar motion and is defined by two displacement variables and one rotation variable. The wrapping connectors ordinarily take the form of a belt or chain. Note that the spheric pair may be viewed as a wrapping pair for its hollow element is "wrapped around" its full element, but it is not a wrapping connector which transmits motion in only one direction. The higher pairs, according to the definition by Reuleaux, are the pairs with the surface elements so shaped that only line or point contacts are possible between elements.

5. Gruebler's Criterion of Mobility

Before a spatial mechanism is established and motion analysis started, the first question is wether or not the mechanism will move. Gruebler's criterion [15] is a quick and easy test for that. It is derived as follows:

Since any rigid body has six degree of freedom in space, n links in space, considered independently, have a total of 6n degree of freedom. If these links were combined to form a mechanism, one link would be taken to be a stationary frame of reference. Thus the total number of degrees of freedom, f, with respect to that stationary frame would become

$$f = 6(n - 1) \qquad (01)$$

when these links are connected by the various pairs, they are imposed by certain restraints on the movability. The total number of degrees of freedom which remains in the mechanism is now

$$f = 6(n - 1) - \sum_{i=1}^{k} r^i \tag{02}$$

where: r^i is the number of restraints imposed by i's pair

k is the total number of pairs in the mechanism

Equation (02) is essentially Gruebler's criterion of movability. It shows the number of degrees of freedom of a mechanism as a function of the number of links and the number of restraints imposed by pairs.

The values of r^i are different for different pairs involved. Those for the lower pairs are listed in Table I.

Table I.

Values of r^i For Lower Pairs

Pair	Symbol	r^i
Revolute	R	5
Prismatic	P	5
Screw	H	5
Cylinder	C	4
Planar	F	3
Spheric	S	3

It should be noted that this criterion has several exceptions. In these cases some of the restraints become redundent and the degrees of freedom may be more than that predicted by equation (02). Some known exceptions are the Bennett and the Bricard mechanism, Hook's joint, this R-C-C-R spatial mechanism, and all planar mechanisms. In the case of the planar mechanisms, another form of the criterion should be used, that is

$$f = 3(n - 1) - \sum_{i=1}^{k} (r^i - 3) \tag{03}$$

The parameters in equation (03) are the same as defined for equation (02).

Usually a mechanism is more interesting if it has one degree of freedom. A single input will then completely determine the motion of the entire mechanism. This condition is known as constrained motion.

In a simple closed-loop chain, composed entirely by the binary* links, k is equal to n and for constrained motion

$$1 = 6(n - 1) - \sum_{i=1}^{n} r^i \tag{04}$$

Since any kind of pair is permitted in a closed-loop chain, $\sum_{i=1}^{n} r^i$ is always less than 5n. This yields the following inequality

$$1 \geqq 6(n - 1) - 5n \geqq n - 6$$

*binary link means a link containing the elements of two pairs that is

$$n \leq 7 \tag{05}$$

This means the number of links in a simple closed-loop chain for a constrained motion can not be more than seven.

B. MATHEMATICAL PRELIMINARIES

Tensor calculus came to prominence with the development of general theory of relativity by Einstein in 1916. It provides the only suitable mathematical language for general discussion of that theory. But actually the tensor calculus is older than that. It was invented by the Itatian mathematicians Ricce and Levi-Civita, showing its applications in geometry and classical mathematical physics in 1900. Thus tensor calculus comes near to being a universal language in mathematical physics. Not only does it enable us to express general equations very compactly, but it also guides us in the selection of physical variables, by indicating, automatically, invariance with respect to the transformation of coordinates.

1. Tensor Terminology

Item	Definition
Range convention:	When a suffix (superscript or subscript) occurs unrepeated in a term, it is understood to take all the values 1, 2,...., N, where N is the number of dimensions of the space.
Summation convention:	When a suffix is repeated in a term, summation with respect to that suffix is understood, the range of summation

being 1,2,....,N.

Transformation matrix: The coefficient of transformation between two coordinate systems are represented into a square form matrix in this thesis, it is denoted by A_{mn}.

Orthogonal coordinates: In a coordinate system, the parametric lines of the coordinates are perpendicular to one another at every point.

Orthogonal transformation: If the coordinate of a point Z_n is transformed from one homogeneous coordinate system to another homogeneous coordinate system by the linear equation

$$Z'_m = A_{mn}Z_n + B_m$$

Since the length is an invariant,

$$dZ'_m dZ'_m = A_{mp}A_{mq}dZ_p dZ_q$$

the determinate of A_{mn} satisfies

$$A_{mp}A_{mq} = \delta_{pq}, \quad \text{or}$$

$$|A_{mn}| = \pm 1$$

Thus this transformation is said to be orthogonal.

Tensor Each number, T_i, of a set of quantities associated with a Cartesian coordinate system, X_i, and with a point, P, is said to be a component of a tensor of the first rank* if the

---

*In general (i.e., when a Cartesian coordinate system quantities transform to any other coordinate system X'_i according to the equation $T'_j = A_{ji}T_i$.

Tensors of higher rank can also be defined. That is, a tensor of the n^{th} rank has n inindices and transforms through its multiplication by n coefficients:

$$T'_{\underbrace{ij\ldots k}_{n}} = A_{ia}A_{jb}\ldots A_{kc}T_{\underbrace{ab\ldots c}_{n}}$$

where i, j, ..., and k are called suffixes.

Cartesian tensor: A tensor which transforms according to the same laws when the coordinates undergo an orthogonal transformation, i.e., when we pass a quantity from one set of homogeneous coordinates to another.

Symmetrical tensor: A tensor with two or more suffixes (both superscripts or both subscripts) that its value of the component is unchanged on interchanging any two of

---

is not specified) a tensor of the first rank, also called a contravariant vector, is defined by the transformation $T'^j = (\partial X'^j/\partial X^i)T^i$. The transformation $T'_j = (\partial X^i/\partial X'^j)T_i$ defines what is called a covariant vector. However, with Cartesian coordinates, $(\partial X'^j/\partial X^i) = (\partial X^i/\partial X'^j) = A_{ji}$, so the distinction between contravariant and covariant vectors is eliminated. Throughout this paper, the indices of tensor operation are written as subscripts only.

these suffixes.

Skew-symmetrical tensor: A tensor which leads to a change of its sign without change of its absolute value if any two of the suffixes are interchanged.

Contraction: For a tensor, such as T_{mrsn}, if we contract by writing the same letter as a superscript and as a subscript, say m = n, the result has the tensor character indicated by the remaining suffixes as B_{rs}.

Permutational symbol, ϵ_{ijk}: A tensor which is both isotropic and completely skew-symmetric; the values of its components are obtained as follows:

$$\epsilon_{ijk} = \begin{cases} 0 & \text{if any two indices have the same value.} \\ 1 & \text{if the values of the indices ijk represent an even permutation* of the sequence 1, 2, 3.} \\ -1 & \text{if the values of the indices ijk} \end{cases}$$

*A permutation of the sequence 1, 2,..., n is even if an even number of interchanges of adjacent integers is required to attain the permutation. Similarly, a permutation is odd if an odd number of interchanges is required. Thus,

$\epsilon_{123} = \epsilon_{312} = \epsilon_{231} = 1$, and $\epsilon_{132} = \epsilon_{321} = \epsilon_{213} = -1$.

represent an odd permutation of the sequence 1, 2, 3.

2. Some Tensor Operations

a) The product of any completely symmetrical tensor, say T_{ij}, and any completely skew-symmetric tensor, say S_{ij}, is

$$T_{ij}S_{ij} = 0$$

b) The relation between δ_{ij} and ϵ_{ijk} is given by

$$\epsilon_{ikj}\epsilon_{mpj} = \delta_{im}\delta_{kp} - \delta_{ip}\delta_{km}$$

This property will be used to great advantage in the application of tensor operations discussed in the Appendix D.

CHAPTER III

DERIVATION OF COORDINATE RELATIONSHIP

This thesis is devoted to the use of tensor calculus in spatial mechanism. The "space" of the spatial mechanism is a Euclidean space of three dimensions. In choosing a system of coordinates, as a general rule it is best to use rectangular Cartesians. The only transformations we have to consider are orthogonal transformations. And the determinant of the transformation is 1, i.e., $|A_{ij}| = 1$.

The equations of spatial mechanics are then oriented in the sense of Cartesian tensors. Figure 1. represents the general diagram of the linkage and relationship of the links to the ground coordinate frame. The ground link, Cc_i, has a length of C units and the direction along a unit vector, c_i. The components of c_i with respect to the ground coordinate frame thus are

$$c_1 = \sin \phi^{\text{IV}} \cos \theta^{\text{IV}},\; c_2 = \sin \phi^{\text{IV}} \sin \theta^{\text{IV}},\; c_3 = \cos \theta^{\text{IV}} \quad (06)$$

where ϕ^{IV} and θ^{IV} represent the polar and azimuthal angles, respectively.

Similarly, the first link, Rr_i, is of a magnitude R and directed along a unit vector r_i which originates at the first joint. The components of r_i can be expressed with respect to either its local coordinate frame or the ground coordinate frame. In this thesis, the first link is always in terms of the polar coordinates with

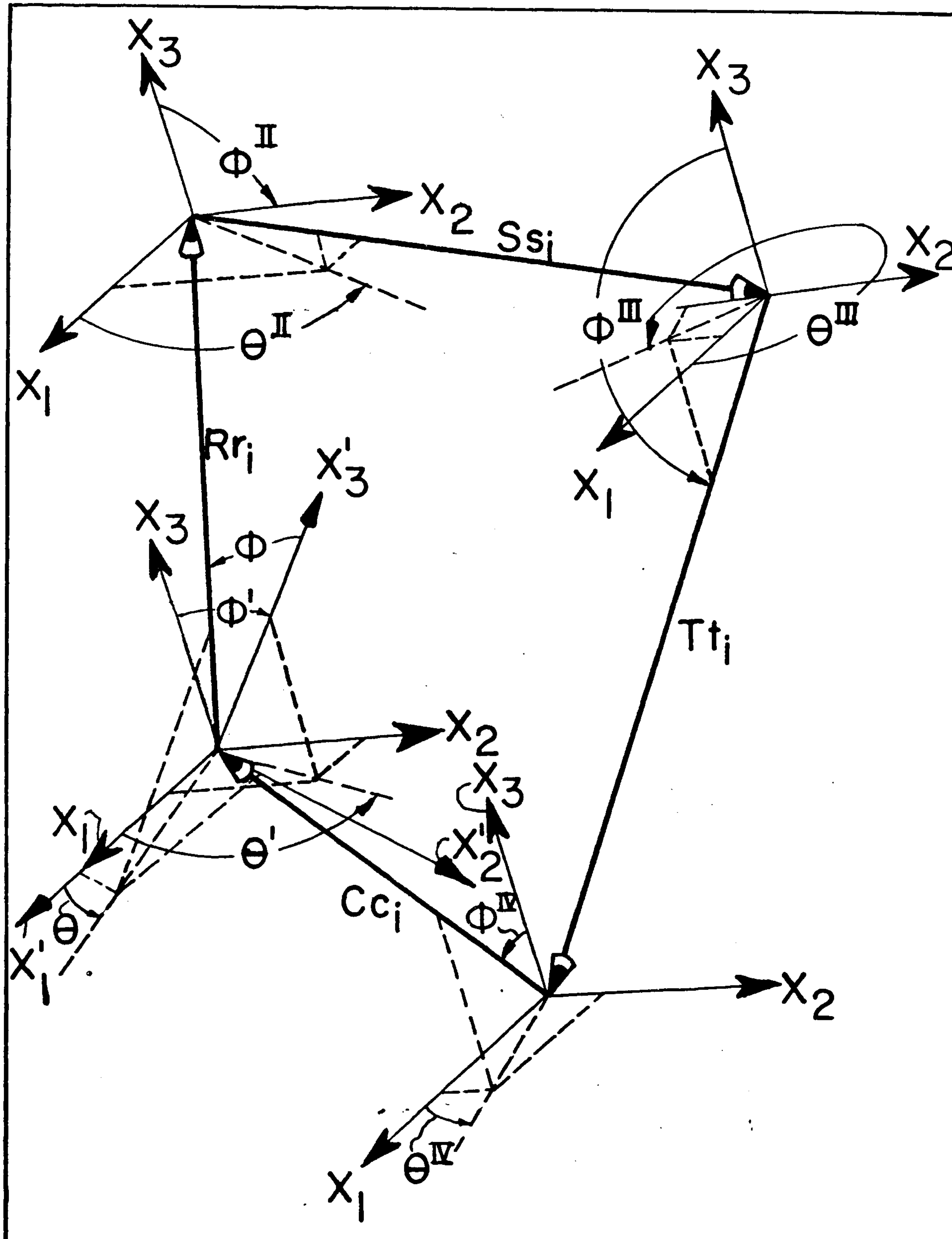

Figure 1. The general diagram of the spatial mechanism

respect to its local coordinate frame as

$$r_1 = \sin \phi \ \cos \theta, \ r_2 = \sin \phi \ \sin \theta, \ r_3 = \cos \phi \qquad (07)$$

where ϕ and θ represent the polar and azimuthal angles respectively in the first local coordinate frame. While the components of the second and third link vectors, or the fourth link vector if any, are expressed in terms of the polar coordinates with respect to the ground coordinate as shown.

The ground coordinate system X_1, X_2, X_3 and the first-joint local coordinate system X_1', X_2', X_3' both take their origins at the ground joint. The transformation matrix A_{ij} can be written as the direction cosine between the axes; that is

$$A_{ij} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

$$\begin{bmatrix} \cos(X_1', X_1) & \cos(X_1', X_2) & \cos(X_1', X_3) \\ \cos(X_2', X_1) & \cos(X_2', X_2) & \cos(X_2', X_3) \\ \cos(X_3', X_1) & \cos(X_3', X_2) & \cos(X_3', X_3) \end{bmatrix}$$

It is convenient that the output link be determined with respect to the ground coordinate frame and the input link to its local coordinate frame by a set of spherical polar coordinates especially when the first joint is revolute. Even though it has one degree of freedom rotating about a fixed axis, it is hard to find directly the relationship between the polar angle and the azimuthal angle of that link with respect to a certain ground reference frame. Therefore it will be well to define a local reference frame in such a way that makes the axis of rotation of the first link (crank) be one of the axes of the local coordinate frame. Whenever one of the axes of the local reference coordinate is specified, the other two axes of the frame system is selected as follows:

Let X_3', usually the axis of rotation or the line of sliding of the first link, be the specified axis of the local frame system in X_1, X_2, X_3 ground reference system with given polar and azimuthal angle $\emptyset'$ and θ' (see Figure 1.), then it can be expressed as

$$(X_3')_i = U_i' \ (\sin \emptyset' \cos \theta', \ \sin \emptyset' \sin \theta', \ \cos \emptyset' \ ) \qquad (08)$$

The other two prime reference axes are defined by

$$(X_1')_i = \frac{\epsilon_{ik3} \ (X_3')_k \ X_3}{\left|\epsilon_{ik3} \ (X_3')_k \ X_3\right|} = \frac{\epsilon_{ik3} \ U_k}{\left|\epsilon_{ik3} \ U_k\right|} \qquad (09)$$

and

$$(X_2')_i = \frac{\epsilon_{ijk} \ (X_3')_j \ (X_1')_k}{\left|\epsilon_{ijk} \ (X_3)_j \ (X_1)_k\right|} = \frac{\epsilon_{ijk} \ U_j \ \epsilon_{km3} \ U_m}{\left|\epsilon_{ijk} \ U_j \ \epsilon_{km3} \ U_m\right|}$$

$$= \frac{(\delta_{im} \ \delta_{j3} - \delta_{i3} \ \delta_{jm}) \ U_j \ U_m}{\left|\epsilon_{ijk} \ U_j \ \epsilon_{km3} \ U_m\right|}$$

$$= \frac{U_3 \ U_i - \delta_{i3}}{\left|U_3 \ U_i - \delta_{i3}\right|} \qquad (10)$$

Equations (09) and (10) can be expanded and formulated to be the unit tensor as below:

$$(X_1')_i = V_i' \; (\sin \theta', -\cos \theta', 0) \tag{11}$$

and

$$(X_2')_i = W_i' \; (\cos \theta' \cos \phi', \sin \theta' \cos \phi', -\sin \phi') \tag{12}$$

From equations (08), (11), and (12), the relation of the local coordinate frame to the ground coordinate system can be readily shown as:

$$(X)_m' = A_{mn} \; (X)_n \tag{13}$$

where

$$A_{mn} = \begin{bmatrix} \sin \theta' & -\cos \theta' & 0 \\ \cos \theta' \cos \phi' & \sin \theta' \cos \phi' & -\sin \phi' \\ \sin \phi' \cos \theta' & \sin \phi' \sin \phi' & \cos \phi' \end{bmatrix}$$

Therefore, the unit vector of the first link, say X_j', expressed in polar coordinates with respect to its local reference frame can be transformed to that with respect to the ground coordinate frame by this relation:

$$X_i = A_{ij} \; X_j' \tag{14}$$

In designing a mechanism, usually two vectors are previously specified rather than the ground coordinate frame. One is related to the input link; the other the output link. For this reason, it will be more convenient to define the ground reference frame as below:

Let X_3, one of the ground coordinates, always coincide with the specified vector which is related to the output link. For example, the axis of rotation of the driven link or the dreven link itself. X_1 is made to be parallel to the X_1' as defined before. Then X_2 is established by cross product of X_3 and X_1 easily.

By doing that, the azimuthal angle θ' of the X_3' can be either 90 or 270 degrees. Therefore the matrix of transformation becomes either

$$A_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \phi' & -\sin \phi' \\ 0 & \sin \phi' & \cos \phi' \end{bmatrix} \tag{15}$$

with the determinat of the transformation equal to 1, or

$$A_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\cos \phi' & -\sin \phi' \\ 0 & -\sin \phi' & \cos \phi' \end{bmatrix} \tag{16}$$

with the determinant of the transformation equal to -1.

While in this paper, only the matrix of transformation with its determinant equal to +1 is considered.

CHAPTER IV

DESCRIPTION AND ANALYSIS OF R-C-R-C-R SPATIAL MECHANISM

It is important to discuss R-C-R-C-R spatial mechanism first, because of its basic nature in spatial motions and relation to the R-C-C-R spatial mechanism. Figure 2. is a diagram of this mechanism. The input link 1 rotates about a fixed axis A-A. Link 2 is a round rod with the first part of it parallel to the axis A-A and connected with link 1 by a cylindrical pair. The second part is bent in a angle equal to the offset angle between the axis A-A and the axis B-B about which the output link 4 rotates. At this part, a revolute pair is made to join link 3. Link 4 again is connected with link 3 by a cylindrical pair.

When the input link 1 is rotating, the output link 4 may be oscillating or rotating depending on its relative position being chosen. It is obvious that the path of every point at the second part of link 2 is an ellipse. Thus the second part of link 2 forms an elliptical cylinder when the input link 1 makes a complete rotation. If the point of intersection P of the center line of link 3 and the axis B-B is constructed within this elliptical cylinder, the output link 4 will be rotating. Otherwise, it will be oscillating about the axis B-B. And the location of the axis B-B also affects the speed-ratio between the driving link and the driven link.

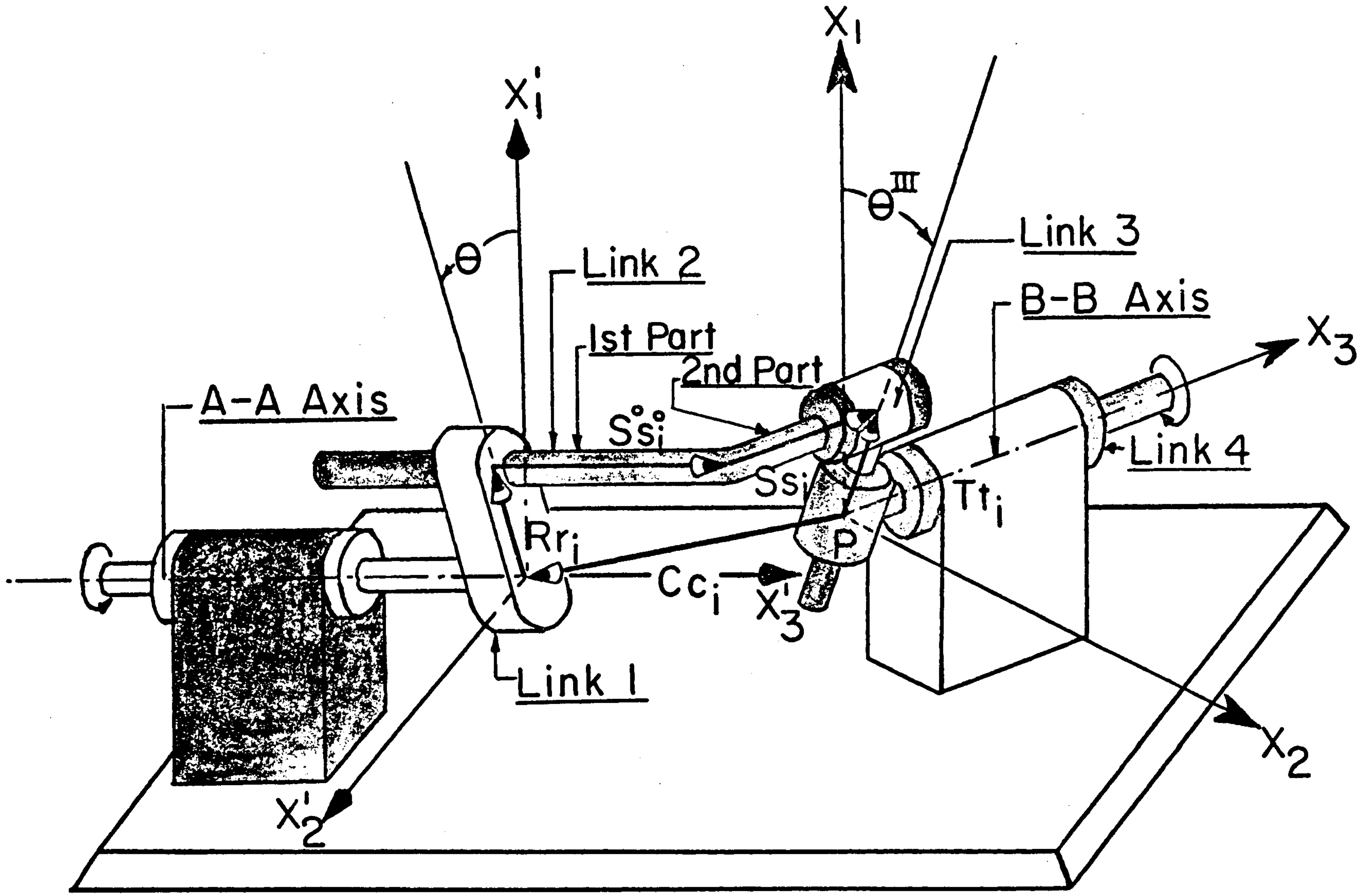

Figure 2. R-C-R-C-R spatial mechanism

In analyzing this mechanism, the main concept is how to choose a suitable coordinate system. Usually the known properties for this mechanism are

a) The ground link Cc_i.

b) The radius of the driving link R.

c) The unit vector of the first part of link 2, s_i^o or the axis A-A.

d) The second part of link 2, Ss_i or the axis B-B.

The input variable is the orientation of the driving link, r_i, with its polar and azimuthal angle ϕ and θ respectively in the first local coordinate frame.

The quantites to be determined are

a) The length of the first part of link 2, S^o.

b) The length T and the unit vector t_i in terms of its polar and azimuthal angle ϕ^{III} and θ^{III} respectively with respect to the ground reference system.

The entire mechanism forms a closed-loop. It can be expressed by a tensor equation as

$$Cc_i + Rr_i + S^o s_i^o + Ss_i - Tt_i = 0 \tag{17}$$

Since the input variable r_i rotates about the axis A-A, hence it has one degree of freedom. If the driving link and the driven link are perpendicular to their axes of rotation. This gives two conditions as

$$r_i s_i^o = 0 \tag{18}$$

and

$$s_i t_i = 0 \tag{19}$$

Let the ground coordinate frame be assigned in the way described in Chapter III. As previously described, the second part of link 2 is always parallel to the axis B-B, and the first part of link 2 is always parallel to the axis A-A. Thus the components of s_i and s_i^o become

$$s_1 = 1, \; s_2 = s_3 = 0 \tag{20}$$

and

$$s_1^o = 0, \; s_2^o = \sin \phi', \; s_3^o = \cos \phi' \tag{21}$$

If the axis A-A is defined by the given polar and azimuthal angle ϕ' and θ' and to be one of the first local reference frame X_3'. The polar angle ϕ of r_i becomes 90 degrees. Then r_i can be denoted with respect to the ground coordinate frame through the transformation matrix (15) and the relation (14) as

$$r_1 = \sin \phi' \cos \theta, \; r_2 = -\cos \phi' \cos \theta, \; r_3 = -\sin \theta \tag{22}$$

From the equation (19), we also have

$$t_3 = \cos \phi^{III} = 0$$

or

$$\phi^{III} = 90^o \tag{23}$$

Assume $K_i = Cc_i + Rr_i$, the equation (17) gives three simultaneous equation as

$$K_1 + S^o s_1^o - Tt_1 = 0 \tag{24}$$

$$K_2 + S^o s_2^o - Tt_2 = 0 \tag{25}$$

$$K_3 + S^o s_3^o + S - Tt_3 = 0 \tag{26}$$

or

$$K_1 + S^o s_1^o - T \cos \theta^{III} = 0 \tag{27}$$

$$K_2 + S^o s_2^o - T \sin \theta^{III} = 0 \tag{28}$$

$$K_3 + S^o s_3^o + S = 0 \tag{29}$$

Solving for S^o, θ^{III}, and T in those three simultaneous equations, we have

$$S^o = -(K_3 + S)/s_3^o \tag{30}$$

$$\theta^{III} = \arctan (E/F) \tag{31}$$

and

$$T = (E^2 + F^2)^{1/2} \tag{32}$$

where

$$E = K_2 - (K_3 + S)s_2^o/s_3^o$$

$$F = K_1 - (K_3 + S)s_1^o/s_3^o$$

The kinematic properties can be readily gotten by differentiating the equations (30) to (32) as

$$\dot{S}^o = -\dot{K}_3/S_3^o \tag{33}$$

$$\dot{\theta}^{III} = (F\dot{E} - E\dot{F})/(E^2 + F^2) \tag{34}$$

$$\dot{T} = (E\dot{E} + F\dot{F})/(E^2 + F^2)^{1/2} \tag{35}$$

$$\ddot{S}^o = -\ddot{K}_3/S_3^o \tag{36}$$

$$\ddot{\theta}^{III} = (F\ddot{E} - E\ddot{F})/(E^2 + F^2) - 2(F\dot{E} - E\dot{F})(E\dot{E} + F\dot{F})/(E^2 + F^2)^2 \tag{37}$$

and

$$\ddot{T} = (E^2 + F^2)^{-\frac{3}{2}}\left[(\dot{E}^2 + \dot{F}^2 + E\ddot{E} + F\ddot{F})(E^2 + F^2) - (E\dot{E} + F\dot{F})^2\right] \tag{38}$$

where

$$\dot{K}_1 = -R\,\dot{\theta}\sin\phi'\sin\theta$$

$$\dot{K}_2 = R\,\dot{\theta}\cos\phi'\sin\theta$$

$$\dot{K}_3 = -R\,\dot{\theta}\cos\theta$$

$$\ddot{K}_1 = -R\sin\phi'(\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta)$$

$$\ddot{K}_2 = R\cos\phi'(\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta)$$

$$\ddot{K}_3 = -R(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta)$$

$$\dot{E} = \dot{K}_2 - \dot{K}_3 s_2^o/s_3^o$$

$$\dot{F} = \dot{K}_1 - \dot{K}_3 s_1^o/s_3^o$$

$$\ddot{E} = \ddot{K}_2 - \ddot{K}_3 s_2^o/s_3^o$$

$$\ddot{F} = \ddot{K}_1 - \ddot{K}_3 s_1^o/s_3^o$$

Where $\dot{\theta}$ and $\ddot{\theta}$ denote the angular velocity and acceleration of the driving link respectively. And $\dot{\theta}^{*}$ and $\ddot{\theta}^{*}$ represents the angular velocity and acceleration of the driven link in the ground coordinate so defined.

By knowing these properties, the dynamical characteristics can be easily analyzed.

The output torque of this mechanism can be obtained by the energy method as follows:

Let TI51 = input torque acting on shaft of link 1

TF51 = couple produced by the first revolute pair due to bearing friction

TF12 = couple produced by the first cylindrical pair due to bearing friction

FF12 = friction force due to the sliding motion of link 2 in the first cylindrical pair

TF23 = couple produced by the second revolute pair due to bearing friction

FF34 = friction force due to the sliding motion of link 3 in the second cylindrical pair

TF45 = couple produced by the third revolute pair due to bearing friction

TO45 = output torque from shaft of link 4

Since the input power must be equal to the sum of the output power and the power dissipated by heat due to friction, hence we have

$$(TI51 - TF51)\dot{\theta} = (TF12 + TF23 + TF45 + TO45)\dot{\theta}^{III} + (FF12)|\dot{S}^{o}| + (FF34)|\dot{T}|$$

where $|\dot{S}^{o}|$ and $|\dot{T}|$ are absolute values formulated in equations (33) and (35) respectively. This may be written as

$$TO45 = (TI51 - TF51)\frac{\dot{\theta}}{\dot{\theta}^{III}} - FF12\frac{|\dot{S}^{o}|}{\dot{\theta}^{III}} - FF34\frac{|\dot{T}|}{\dot{\theta}^{III}} - TF12 - TF23 - TF45 \qquad (39)$$

A nearly smooth transmission can be obtained for this mechanism, if the axis B-B and the axis A-A are on the same plane.

The condition for two axes on the same plane is $\theta^{IV} = 270^{\circ}$. When the driving link and the driven link are at their starting positions, they should both be parallel to the X_1 or X_1' coordinate, i.e., $\theta = \theta^{III} = 0$, equations (27), (28), and (29) can be rewritten as

$$R - T = 0 \tag{40}$$

$$C \sin \phi^{IV} + S^{o} \sin \phi' = 0 \tag{41}$$

$$C \cos \phi^{IV} + S^{o} \cos \phi' - S = 0 \tag{42}$$

Equation (40) just indicates that the input link and the output link are equal in length at this moment.

To eliminate S^{o} from equations (41) and (42) gives a critical condition for S,

$$S = C(\sin \phi^{IV} \cos \phi'/\sin \phi' - \cos \phi\ ) \tag{43}$$

If $\sin \phi^{IV}$ is made equal to $-\sin \phi'$, then

$$\phi^{IV} = 180^{\circ} - \phi' \qquad (\text{for } 0^{\circ} \leq \phi^{IV} \leq 180^{\circ})$$

this gives S = 0 in equation (43). It means that the point P is also located on the A-A axis.

Numerical Example:

Given a R-C-R-C-R spatial mechanism for which $C = 2.5''$, $R = 1''$, and $\phi' = 150^{\circ}$. Assume the couple due to friction by each bearing connection of the revolute parts be 0.01 in-lb and the friction force due to sliding in each cylindrical pair be 0.03 lb. For a constant input rotating speed of 10 rad/sec and input torque of 1 in-lb, the pair variables are plotted through computer program shown in Appendix I. Output angle, output angular velocity and acceleration, and output torque are shown in figure 3. to figure 6. respectively. The other variables are show in figure 18. to figure 23. in Appendix J.

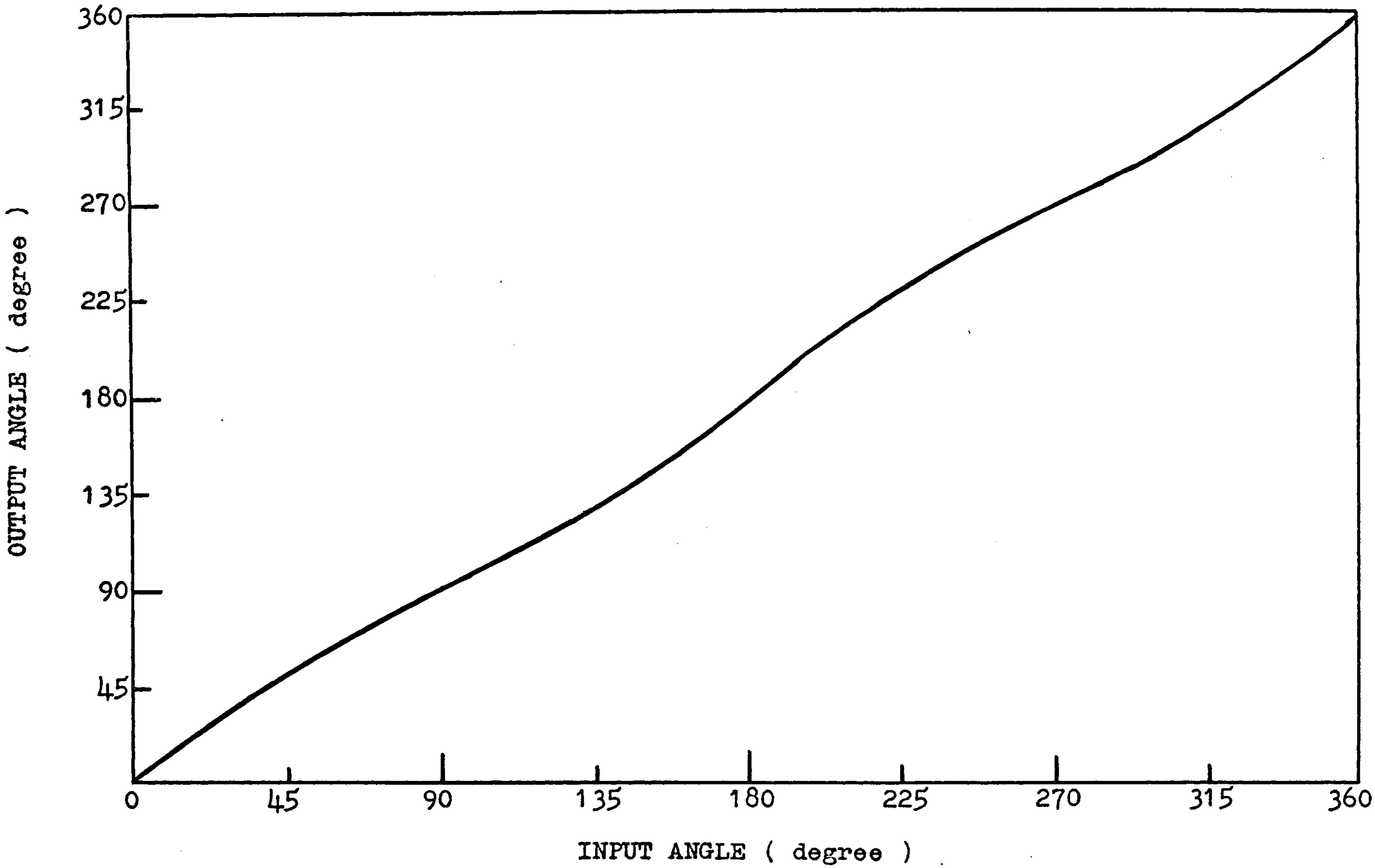

Figure 3. Output angle versus θ

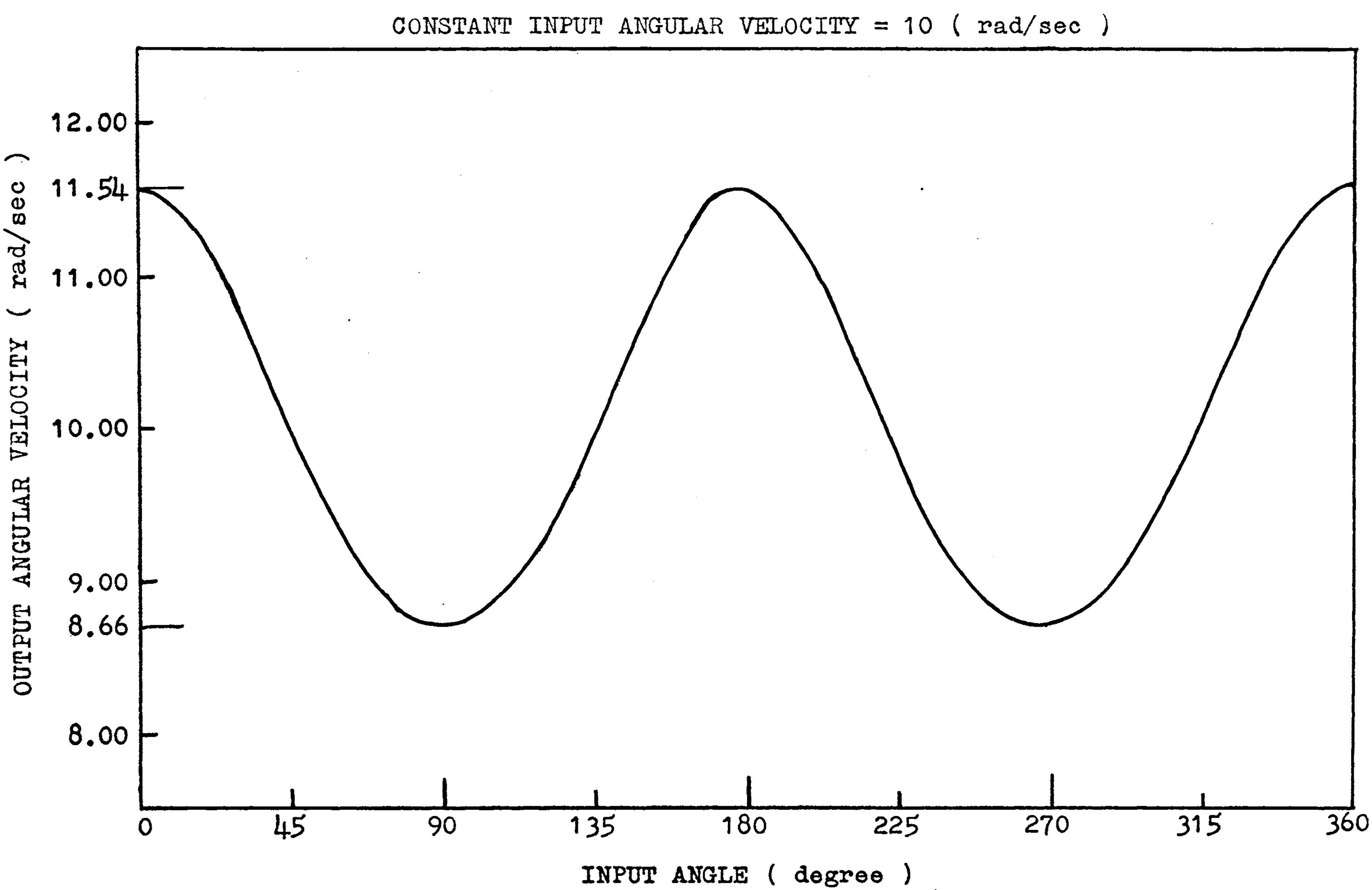

Figure 4. Output angular velocity versus θ

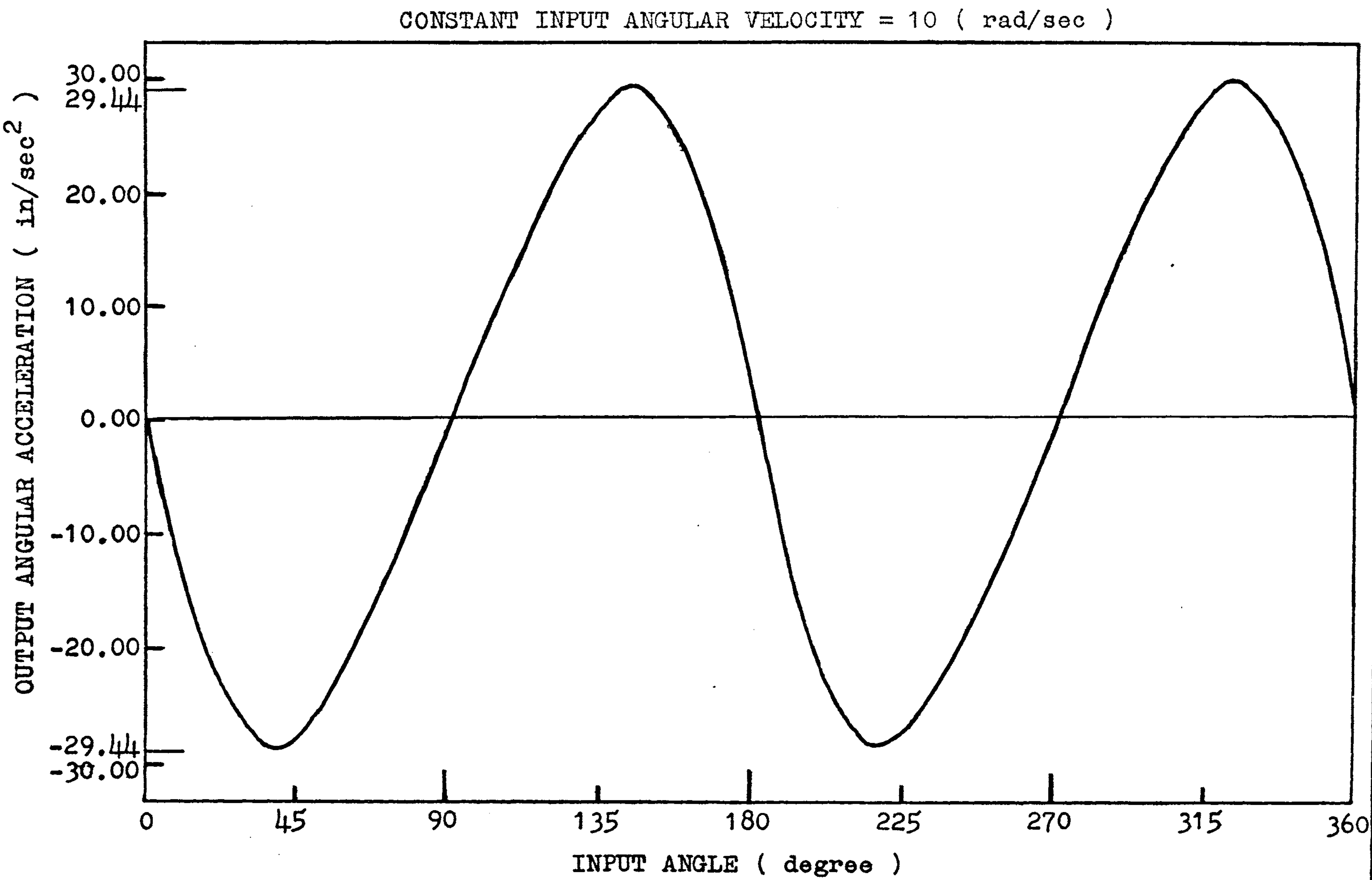

Figure 5. Output angular acceleration versus θ

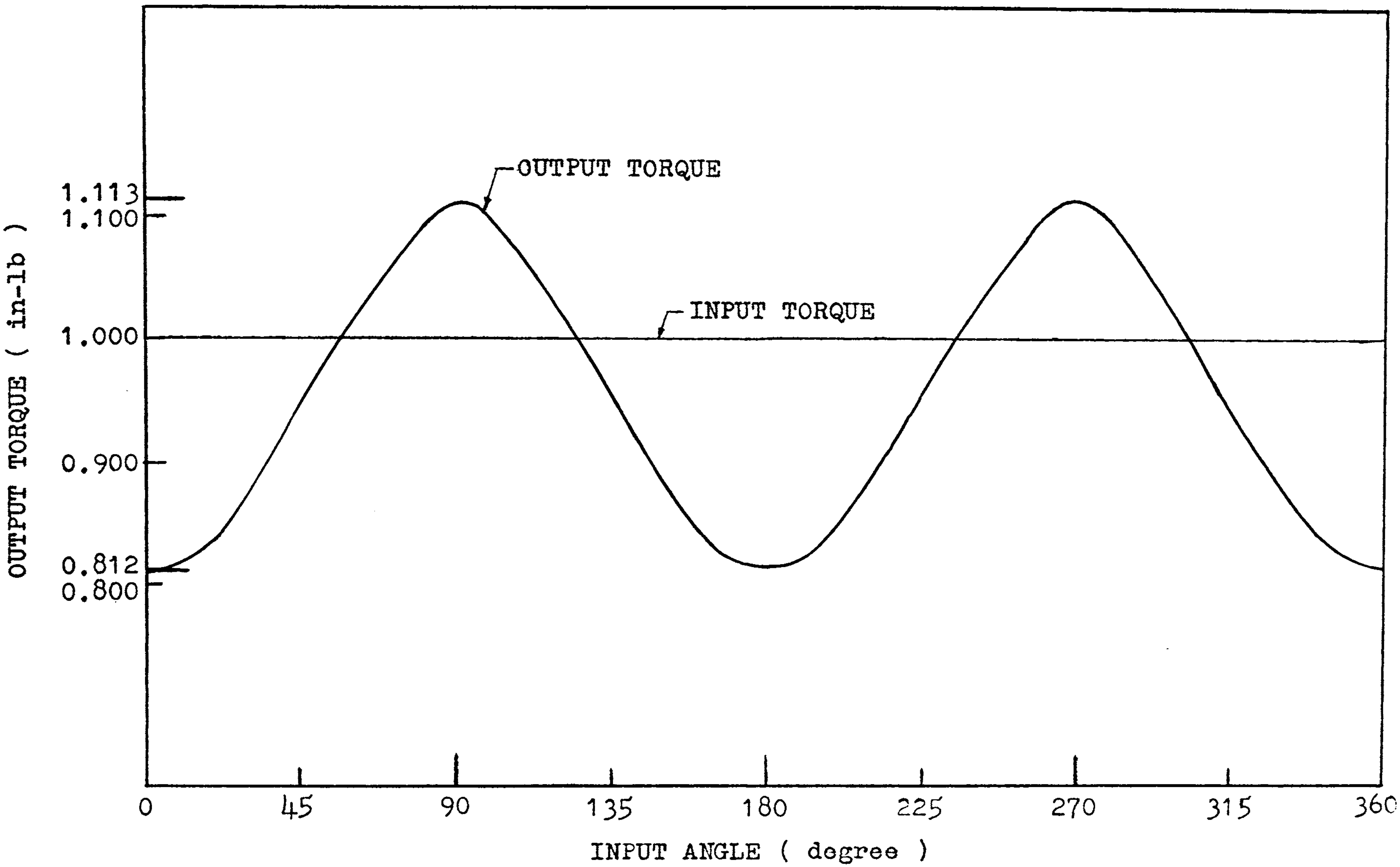

Figure 6. Output torque versus θ for a constant input torque of 1 in-lb

CHAPTER V

DESIGN OF R-C-C-R SPATIAL MECHANISM

A. Construction of R-C-C-R Spatial Mechanism

In reality, R-C-C-R spatial mechanism is a combination form of two symmetrical R-C-R-C-R spatial mechanisms discussed in Chapter IV. Due to their symmetry, the first parts of link 2 of two R-C-R-C-R mechanisms can be connected directly as shown in Figure 7. The input shaft and the output shaft are on the same plane, intersecting at a skew angle and separated by a distance C which forms the ground link in this mechanism. Two cylinders are installed on the top of the driving crank and the driven crank of equal length. Their center lines are parallel to their shafts respectively. A round connecting rod is bent in an angle which is equal to the skew angle of the shafts intersection and inserted into two cylinders. Thus a complete R-C-C-R mechanism with symmetry is constructed.

B. Derivation of the Governing Equations

According to Gruebler's criterion of movability, R-C-C-R spatial mechanism should be of zero degrees of freedom and not movable. But actually its degree of freedom is one. The exception is due to the fact that the center lines of two cylinders are intersected exactly in the same angle as the shafts offsetted. This makes one of the restraint equations redundant.

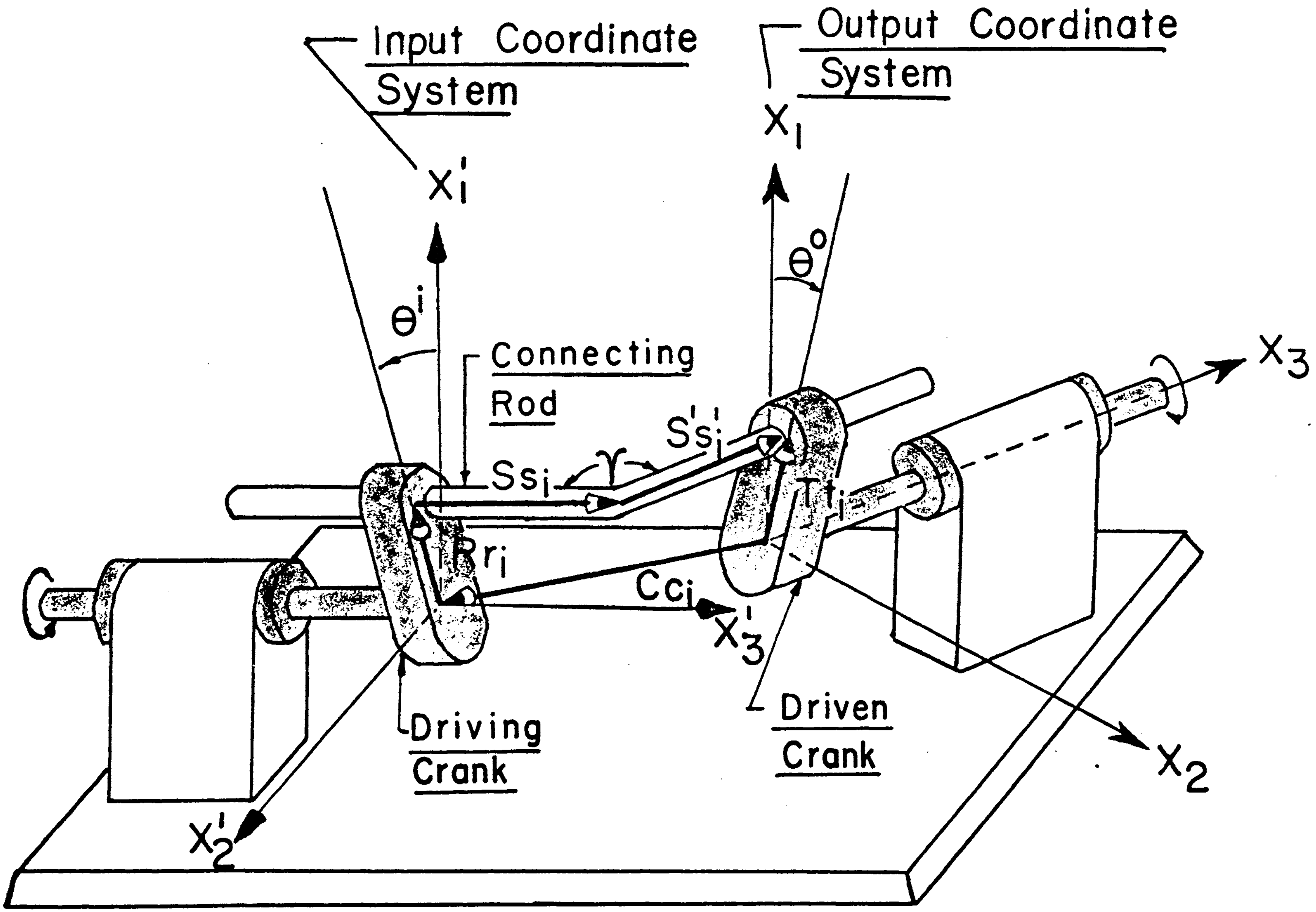

Figure 7. R-C-C-R spatial mechanism

The closed-loop vector equation used here is

$$Cc_i + Rr_i + Ss_i + S's_i' - Tt_i = 0 \tag{44}$$

if the direction of the vectors are so defined in Figure 7. The components of each vector in terms of their polar and azimuthal angles with respect to the output coordinate system are:

c_i : $(\sin \phi^{III} \cos \theta^{III}, \sin \phi^{III} \sin \theta^{III}, \cos \phi^{III})$

r_i : $A_{ij}\, r_j(\cos \theta^{i}, \sin \theta^{i}, 0)$

where A_{ij} is the matrix of transformation for the input coordinate system and r_j is in terms of the input coordinate system.

s_i : $(\sin \phi^{II} \cos \theta^{II}, \sin \phi^{II} \sin \theta^{II}, \cos \phi^{II})$

s_i' : $(\sin \phi^{II'} \cos \theta^{II'}, \sin \phi^{II'} \sin \theta^{II'}, \cos \phi^{II'})$

and

t_i : $(\cos \theta^{o}, \sin \theta^{o}, 0)$

where θ^{i} indicates the input angle and θ^{o} indicates the output angle with respect to the input coordinate system and output coordinate system respectively.

Let X_3 , one of the ground coordinate system, be coincided with the driven shaft and X_3' be coincided with the driving shaft. Let γ be the bending angle of link 2. By making $\phi' = 180 - \gamma$, the matrix of transformation shown in equation (15) can be written for the input coordinate system as

$$A_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\cos\gamma & -\sin\gamma \\ 0 & \sin\gamma & -\cos\gamma \end{bmatrix} \tag{45}$$

Due to the symmetry and the coordinate relation so defined, this mechanism has several properties as follows:

a) ϕ^{IV}, the polar angle of the ground link Cc_i, equals $180 - \frac{\phi'}{2}$, or $90 + \frac{\gamma}{2}$.

b) θ^{IV}, the azimuthal angle of Cc_i, equals 270 degrees. It is because two shafts are on the same plane.

c) T is equal to R.

d) S' is equal to S.

e) s_i is the same as X_3'.

f) s_i' is the same as X_3.

Thus the components of each vector can be explicitly expressed as the following:

c_i : $(0, \cos\frac{\gamma}{2}, -\sin\frac{\gamma}{2})$

r_i : $(\cos\theta^i, -\cos\gamma \sin\theta^i, \sin\gamma \sin\theta^i)$

s_i : $(0, \sin\gamma, -\cos\gamma)$

s_i' : $(0, 0, 1)$

and

t_i : $(\cos\theta^o, \sin\theta^o, 0)$

Then the closed-loop equation (44) can be expanded into three simultaneous equations

$$R\cos\theta^i - R\cos\theta^o = 0 \tag{46}$$

$$C\cos\frac{\gamma}{2} - R\cos\gamma\sin\theta^i - S\sin\gamma - R\sin\theta^o = 0 \tag{47}$$

$$-C\sin\frac{\gamma}{2} - R\sin\gamma\sin\theta^i - S\cos\gamma - S = 0 \tag{48}$$

Equation (46), clearly indicates that the input angle and the output angle are equal. This fact confirms that the mechanism is able to connect the input shaft and the output shaft at a one-to-one speed ratio. Now replacing θ^i and θ^o by θ in the equations (47) and (48), an identical result is obtained and can be written as

$$S = (C/2 \sin\frac{\gamma}{2}) - R \sin\theta \cot\frac{\gamma}{2} \tag{49}$$

Equation (49) expresses the sliding length of connecting rod as a function of the bending angle γ, the rotating angle θ, and length of the driving or driven link R and the length C of the ground link.

When $\theta = 270^o$, S is at the shortest position. S_{min} can be calculated according to

$$S_{min} = (\frac{C}{2} - R \cos\frac{\gamma}{2})/\sin\frac{\gamma}{2} \tag{50}$$

and when $\theta = 90^o$, where S is at the longest position, S_{max} can be calculated according to

$$S_{max} = (\frac{C}{2} + R \cos\frac{\gamma}{2})/\sin\frac{\gamma}{2} \tag{51}$$

The sliding displacement D of the sliding pair is the distance between the two extreme positions of S_{max} and S_{min}

$$D = S_{max} - S_{min} = 2R \cot\frac{\gamma}{2} \tag{52}$$

Table II is the computer tabulated S_{min} in terms of C at the various bending angles and the R/C ratios.

C. Example for the Design Procedures

Figure 8. shows a pictorial model of the mechanism.

TABLE II

MINIMUN S IN TERMS OF C

GAMMA	R=0.50C	R=0.45C	R=0.40C	R=0.35C	R=0.30C	R=0.25C	R=0.20C	R=0.15C	R=0.10C	R=0.05C
30.0	0.06583	0.25243	0.43903	0.62563	0.81224	0.99884	1.18544	1.37204	1.55865	1.74525
35.0	0.07696	0.23554	0.39412	0.55270	0.71128	0.86986	1.02844	1.18702	1.34560	1.50418
40.0	0.08816	0.22554	0.36291	0.50029	0.63766	0.77503	0.91241	1.04978	1.18715	1.32453
45.0	0.09946	0.22017	0.34088	0.46159	0.58230	0.70301	0.82372	0.94443	1.06514	1.18585
50.0	0.11085	0.21807	0.32530	0.43252	0.53975	0.64697	0.75420	0.86142	0.96865	1.07588
55.0	0.12235	0.21840	0.31445	0.41050	0.50655	0.60259	0.69864	0.79469	0.89074	0.98679
60.0	0.13397	0.22058	0.30718	0.39378	0.48038	0.56699	0.65359	0.74019	0.82680	0.91340
65.0	0.14574	0.22422	0.30271	0.38119	0.45967	0.53816	0.61664	0.69513	0.77361	0.85210
70.0	0.15765	0.22906	0.30046	0.37187	0.44328	0.51469	0.58609	0.65750	0.72891	0.80032
75.0	0.16973	0.23489	0.30005	0.36521	0.43037	0.49553	0.56069	0.62586	0.69102	0.75618
80.0	0.18199	0.24157	0.30116	0.36075	0.42034	0.47992	0.53951	0.59910	0.65869	0.71827
85.0	0.19444	0.24900	0.30357	0.35814	0.41270	0.46727	0.52183	0.57640	0.63096	0.68553
90.0	0.20711	0.25711	0.30711	0.35711	0.40711	0.45711	0.50711	0.55711	0.60711	0.65711
95.0	0.22001	0.26582	0.31164	0.35745	0.40327	0.44909	0.49490	0.54072	0.58654	0.63235
100.0	0.23315	0.27511	0.31706	0.35902	0.40097	0.44293	0.48488	0.52684	0.56879	0.61075
105.0	0.24657	0.28494	0.32331	0.36167	0.40004	0.43840	0.47677	0.51514	0.55350	0.59187
110.0	0.26028	0.29529	0.33030	0.36531	0.40033	0.43534	0.47035	0.50536	0.54037	0.57538
115.0	0.27431	0.30616	0.33802	0.36987	0.40172	0.43358	0.46543	0.49728	0.52914	0.56099
120.0	0.28868	0.31754	0.34641	0.37528	0.40415	0.43301	0.46188	0.49075	0.51962	0.54848
125.0	0.30341	0.32944	0.35546	0.38149	0.40752	0.43355	0.45958	0.48561	0.51163	0.53766
130.0	0.31853	0.34185	0.36517	0.38848	0.41180	0.43511	0.45843	0.48174	0.50506	0.52837
135.0	0.33409	0.35480	0.37551	0.39622	0.41693	0.43764	0.45835	0.47906	0.49977	0.52049
140.0	0.35010	0.36830	0.38650	0.40470	0.42290	0.44110	0.45929	0.47749	0.49569	0.51389
145.0	0.36661	0.38238	0.39814	0.41391	0.42967	0.44544	0.46120	0.47697	0.49273	0.50850
150.0	0.38366	0.39706	0.41046	0.42386	0.43725	0.45065	0.46405	0.47745	0.49084	0.50424
155.0	0.40129	0.41238	0.42346	0.43455	0.44563	0.45672	0.46780	0.47889	0.48997	0.50106
160.0	0.41955	0.42837	0.43718	0.44600	0.45482	0.46363	0.47245	0.48126	0.49008	0.49890
165.0	0.43849	0.44507	0.45165	0.45824	0.46482	0.47140	0.47798	0.48457	0.49115	0.49773
170.0	0.45817	0.46254	0.46691	0.47129	0.47566	0.48004	0.48441	0.48879	0.49316	0.49754
175.0	0.47865	0.48083	0.48301	0.48519	0.48738	0.48956	0.49170	0.49393	0.49611	0.49829
180.0	0.50000	0.50000	0.50000	0.50000	0.50000	0.50000	0.50000	0.50000	0.50000	0.50000

Figure 8. Model of R-C-C-R spatial mechanism

This model will be able to transmit motion at a right angle. The design procedures are according to the following steps:

a) To measure the length of the ground link C, say 2".

b) To decide the length R of the driving or driven crank, say 1".

c) To evaluate the ratio R/C = .5.

d) To measure the angle of the shafts intersection, or the bending angle of the connecting rod , say 90°.

e) To use the table II, the minimum length of the connecting rod S_{min} = .20711 2" = .41422".

f) To use the equation (52), the sliding displacement D = 2" is found, thus the length of the connecting rod S should be greater than $S_{max} = S_{min} + D$ = .41422" + 2" = 2.41422".

CHAPTER VI

CONCLUSION

The simple construction of the R-C-C-R spatial mechanism makes it very attractive. In designing such a device, it is preferable to have an even number of the connecting links symmetrically arranged. In this manner dynamic stability is assured and smooth transmission is maintained. Figure 8. shows a model of this mechanism which has six individual connecting links arranged at 60° intervals.

Figure 9. shows a modified R-C-C-R mechanism, namely R-C-R-C-R mechanism. The bend in the connecting link is accomplished by a hinge joint. Thus, it may be used to couple shafts of varying intersection angles. Furthermore, this construction avoids the manufacturing difficulties of bending the connecting link to the required tolerance.

The R-C-R-C-R mechanism offers merit over Hook's joints and bevel gears for the purpose of indirect transmission, because the transmission torque is not dependent on the rotating angle as Hook's joints and is evenly distributed on the connecting links instead of overloading a single gear tooth pair of the bevel gears.

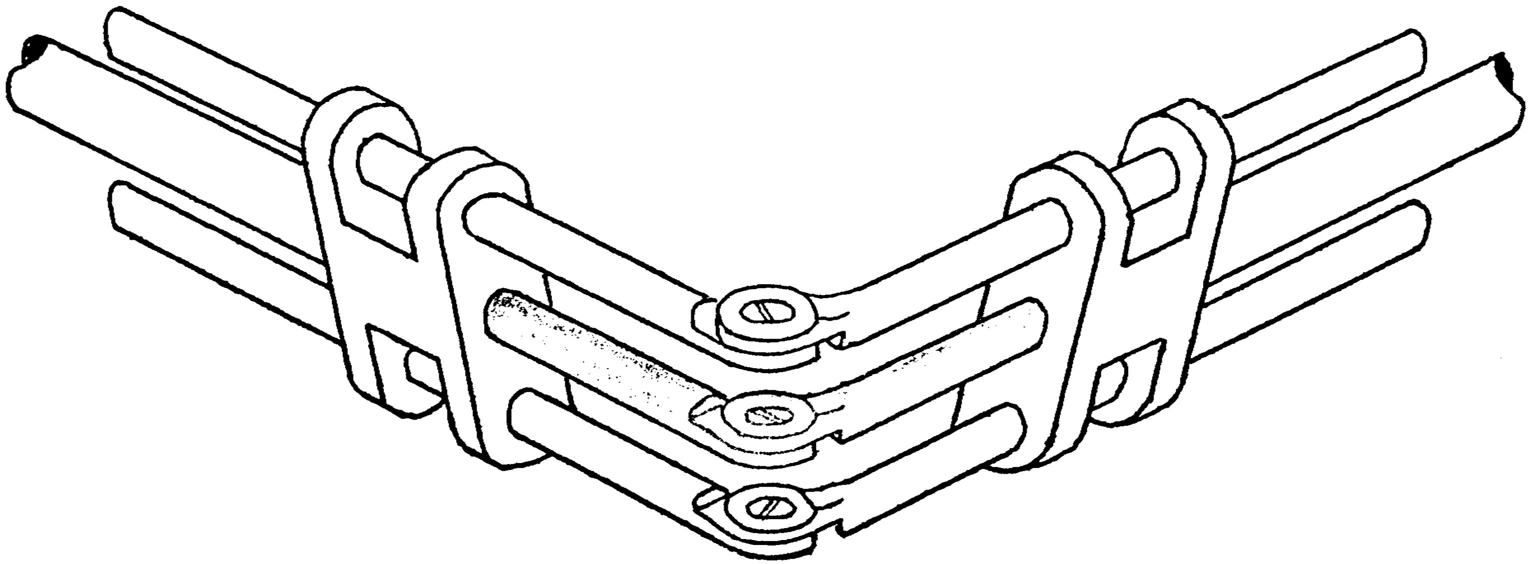

Figure 9. R-C-R-C-R spatial mechanism

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APPENDIX

APPENDIX A

R-S-S-P SPATIAL MECHANISM

Unknowns: ϕ^{II}, θ^{II}, and T

Inputs: r_i

Constants: C, c_i, R, S, and t_i

Loop Equation: $Cc_i + Rr_i + Ss_i + Tt_i = 0$

Let $K_i = Cc_i + Rr_i$, thus

$$K_i + Ss_i + Tt_i = 0 \tag{A-1}$$

If the ground coordinate frame is chosen in such a way that makes $X_3 = t_3$, i.e., $\phi^{III} = 0$, thus

$$t_1 = t_2 = 0, \; t_3 = 1$$

and equation (A-1) becomes

$$K_1 + S \sin \phi^{II} \cos \theta^{II} = 0 \tag{A-2}$$

$$K_2 + S \sin \phi^{II} \sin \theta^{II} = 0 \tag{A-3}$$

$$K_3 + S \cos \phi^{II} + T = 0 \tag{A-4}$$

From equations (A-2) and (A-3), we have

$$\tan \theta^{II} = \frac{K_2}{K_1}$$

or

$$\theta^{II} = \text{arc tan } \left(\frac{K_2}{K_1}\right) \tag{A-5}$$

and

$$\sin \phi^{II} = \frac{(K_1^2 + K_2^2)^{\frac{1}{2}}}{S} \quad \& \quad \cos \phi^{II} = \frac{\pm\left[S^2 - (K_1^2 + K_2^2)\right]^{\frac{1}{2}}}{S}.$$

or

$$\phi^{II} = \text{arc cos } \left\{\frac{\left[S^2 - (K_1^2 + K_2^2)\right]^{\frac{1}{2}}}{S}\right\}$$

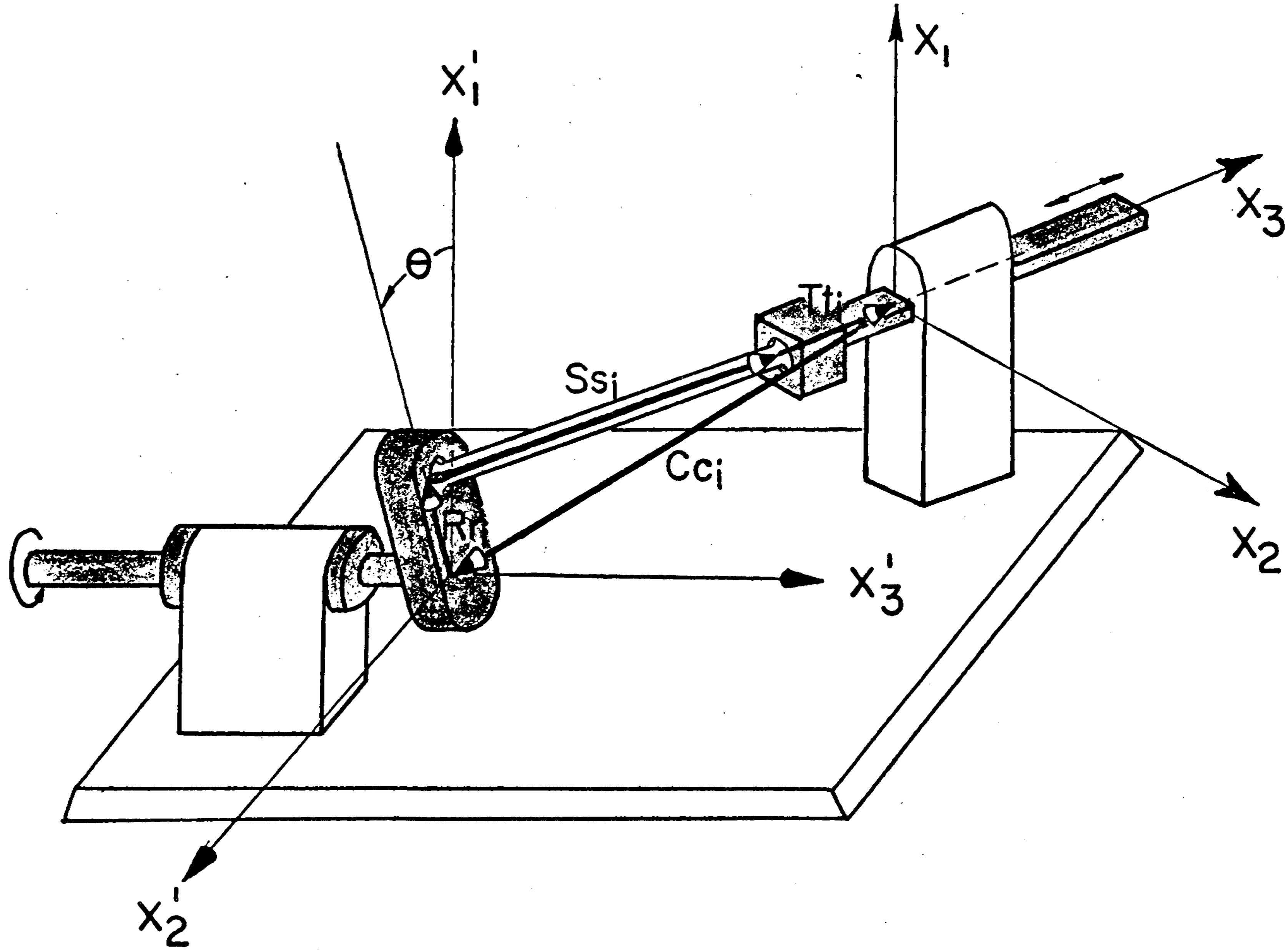

Figure 10. R-S-S-P spatial mechanism

or

$$\pi - \text{arc cos}\left\{\frac{\left[S^2 - (K_1^2 - K_2^2)\right]^{\frac{1}{2}}}{S}\right\} \tag{A-6}$$

Substituting equation (A-6) into equation (A-4) we have

$$T = -K_3 \pm \left[S^2 - (K_1^2 + K_2^2)^{\frac{1}{2}}\right] \tag{A-7}$$

Note: The "+" and "-" signs in equation (A-7) mean there are two ways to be chosen for the linkage. The selection is arbitrary. It depends on which is more suitable to the design requirement. Figure A. shows the positive sign has been chosen.

APPENDIX B

R-S-C-C SPATIAL MECHANISM

Unknowns: S, ϕ^{II}, θ^{II}, and T

Input: r_i

Constants: C, c_i, R, and t_i

Constraint Equation: $s_i t_i = P$

Loop Equation: $Cc_i + Rr_i + Ss_i + Tt_i = 0$

Let $K_i = Cc_i + Rr_i$, the loop equation becomes

$$K_i + Ss_i + Tt_i = 0 \tag{B-1}$$

Multiplying equation (B-1) by t_i, we have

$$T = -(K_i t_i + PS) \tag{B-2}$$

Substituting this into equation (B-1) and expand

$$K_1 + S \sin \phi^{II} \cos \theta^{II} - (K_i t_i + PS)t_1 = 0 \tag{B-3}$$

$$K_2 + S \sin \phi^{II} \sin \theta^{II} - (K_i t_i + PS)t_2 = 0 \tag{B-4}$$

$$K_3 + S \cos \phi^{II} - (K_i t_i + PS)t_3 = 0 \tag{B-5}$$

If the ground coordinate frame is chosen in such a way that gives

$$t_1 = t_2 = 0, \text{ and } t_3 = 1$$

From the constraint equation we have

$$s_3 = P = \cos \phi^{II}$$

or

$$\phi^{II} = \text{arc cos } (P) = \text{arc sin} \left[(1 - P^2)^{\frac{1}{2}}\right]$$

Thus equation (B-3) and (B-4) becomes

$$K_1 + S(1 - P^2)^{\frac{1}{2}} \cos \theta^{II} = 0 \tag{B-6}$$

$$K_2 + S(1 - P^2)^{\frac{1}{2}} \sin \theta^{II} = 0 \tag{B-7}$$

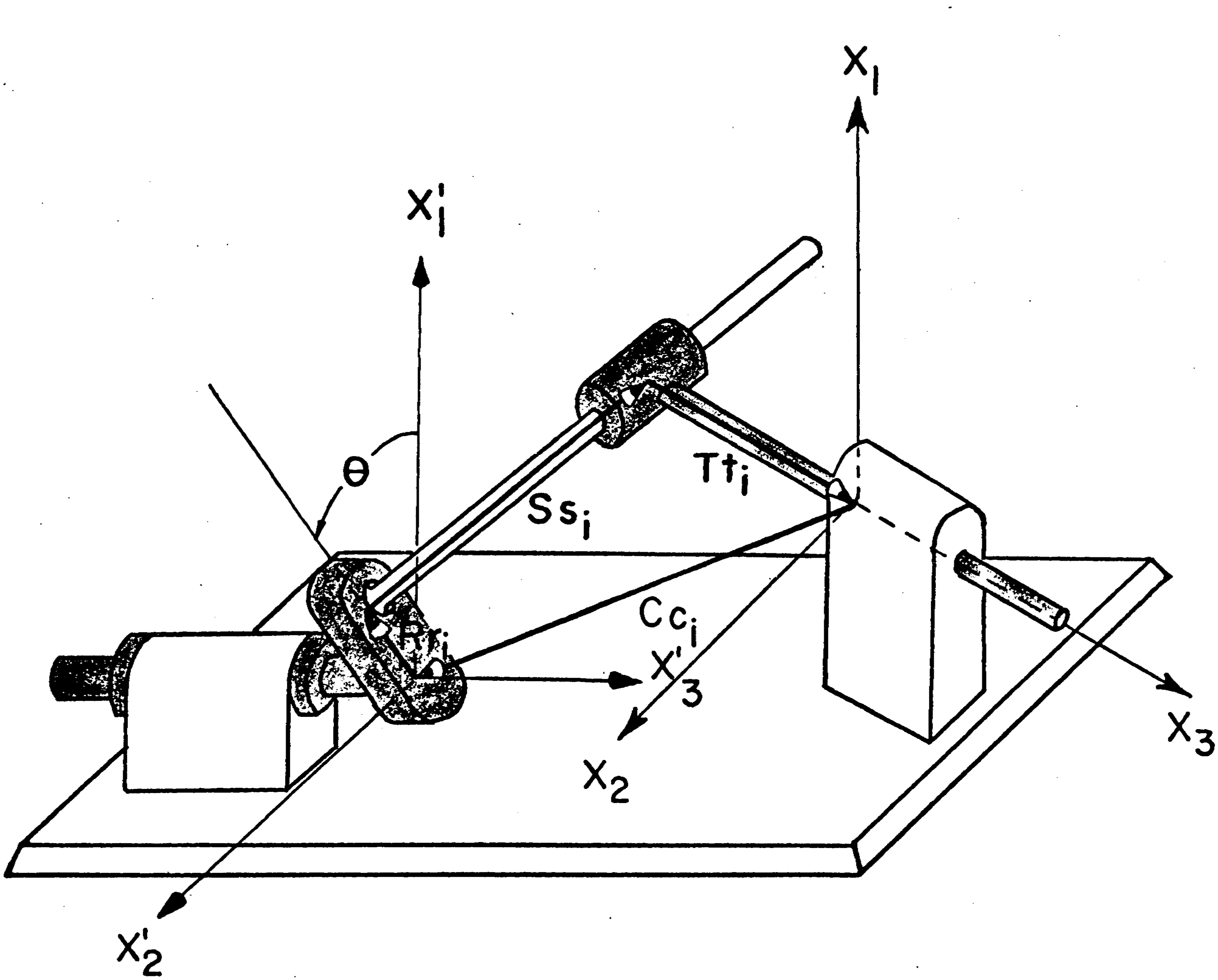

Figure 11. R-S-C-C spatial mechanism

Solving for θ^{II}, and S, we have

$$\theta^{II} = \arc\tan\left(\frac{K_2}{K_1}\right) \tag{B-8}$$

$$S = \left(\frac{K_1^2 + K_2^2}{1 - P^2}\right)^{\frac{1}{2}} \tag{B-9}$$

Substituting S into equation (B-2) and solving for T

$$T = -\left[K_3 + P\left(\frac{K_1^2 + K_2^2}{1 - P^2}\right)^{\frac{1}{2}}\right] \tag{B-10}$$

APPENDIX C

R-S-C-P SPATIAL MECHANISM

Unknowns: S, $\phi^{o^{II}}$, $\theta^{o^{II}}$, and T

Inputs: r_i

Constants: C, c_i, R, S^o, and s_i

Constraint Equation: $s_i^o s_i = P$

$$s_i t_i = Q \tag{C-0}$$

Loop Equation: $Cc_i + Rr_i + S^o s_i^o + Ss_i + Tt_i = 0$

Let $K_i = Cc_i + Rr_i$, the closed-loop equation becomes

$$K_i + S^o s_i^o + Ss_i + Tt_i = 0 \tag{C-1}$$

Multiplying equation (C-1) by s_i and from the constraint we have

$$K_i s_i + S^o P + S + T\, s_i t_i = 0$$

or

$$S = -(K_i s_i + S^o P + TQ) \tag{C-2}$$

Substituting equation (C-2) into equation (C-1)

$$K_i + S^o s_i^o - (K_i s_i + S^o P + TQ)s_i + Tt_i = 0 \tag{C-3}$$

If we choose the ground coordinate frame in such a way that makes $s_1 = s_2 = 0$, $s_3 = 1$ and let X_1 be perpendicular to t_i, i.e., $\theta^{II} = 270^o$.

Then from equation (C-0), we get

$$t_i = 0,\ t_2 = -(1 - Q^2)^{\frac{1}{2}},\ \text{and } t_3 = Q$$

Thus equation (C-3) can be expanded as

$$K_1 + S^o \sin \phi^{o^{II}} \cos \theta^{o^{II}} = 0 \tag{C-4}$$

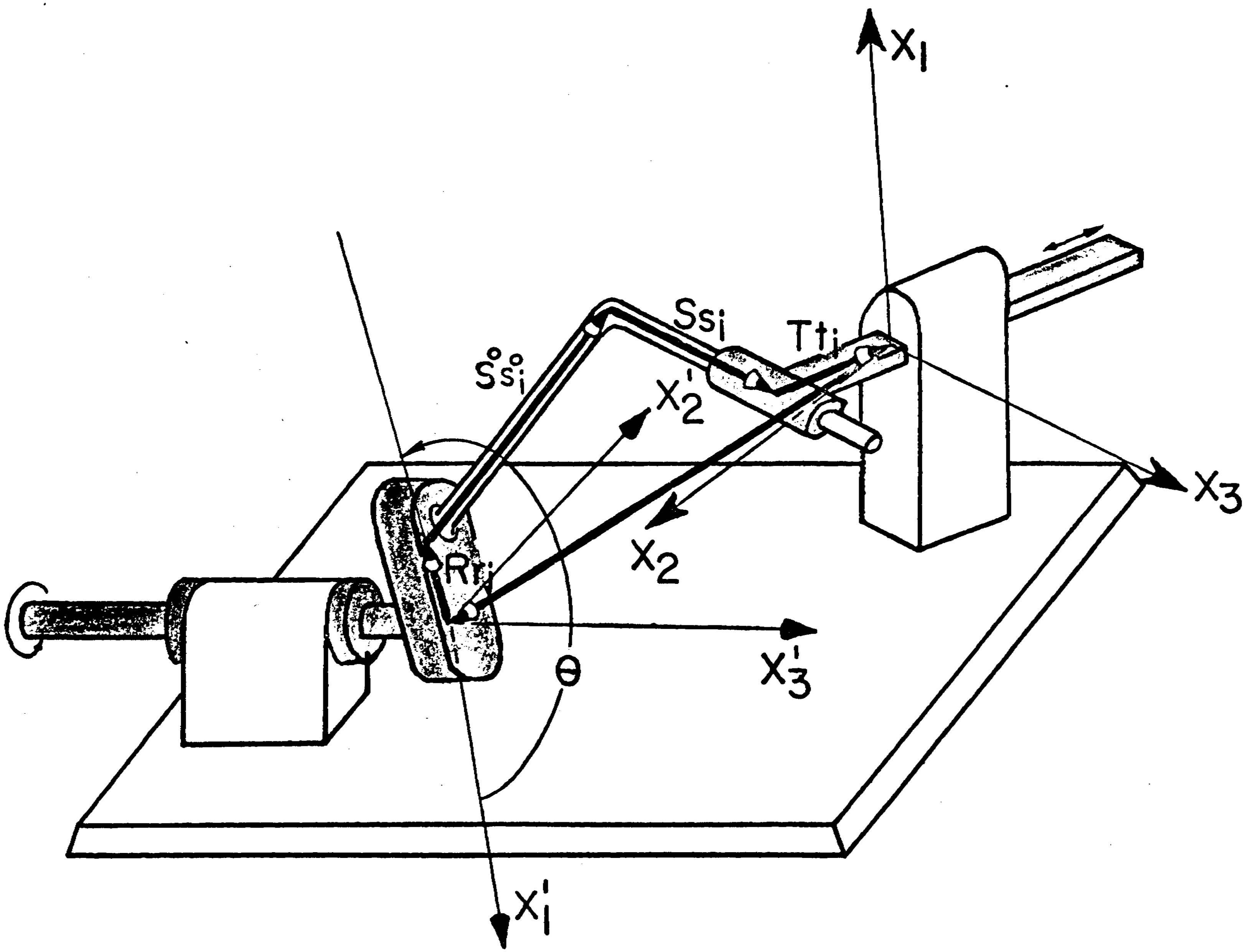

Figure 12. R-S-C-P spatial mechanism

$$K_2 + S^o \sin \phi^{o^{II}} \sin \theta^{o^{II}} - T(1 - Q^2)^{\frac{1}{2}} = 0 \qquad (C\text{-}5)$$

$$S^o(\cos \phi^{o^{II}} - P) = 0 \qquad (C\text{-}6)$$

Equation (C-6) gives

$$s_3^o = \cos \phi^{o^{II}} = P$$

or

$$\phi^{o^{II}} = \text{arc cos } (P) = \text{arc sin}\left[(1 - P^2)^{\frac{1}{2}}\right] \qquad (C\text{-}7)$$

Substituting this into equation (C-4) and solving for $\theta^{o^{II}}$,

$$\theta^{o^{II}} = \text{arc cos } \left(- \frac{K_1}{S^o(1 - P^2)^{\frac{1}{2}}}\right) \qquad (C\text{-}8)$$

From equation (C-5), we have

$$T = \frac{1}{(1 - Q^2)^{\frac{1}{2}}} \left[K_2 + \left\{S^{o^2}(1 - P^2) - K_1^2\right\}^{\frac{1}{2}}\right] \qquad (C\text{-}9)$$

Substituting back into equation (C-2),

$$S = - K_3 + S^oP + \frac{Q}{(1 - Q^2)^{\frac{1}{2}}}\left[K_2 + \left\{S^{o^2}(1 - P^2) - K_1^2\right\}^{\frac{1}{2}}\right] \qquad (C\text{-}10)$$

To get a rational solution from equation (C-8), we know the following condition must be fulfilled,

$$S^o \geq \frac{|Cc_1 + Rr_1|}{1 - P^2} \qquad (C\text{-}11)$$

APPENDIX D

H-C-C-C SPATIAL MECHANISM

Unknowns: S, T, ϕ^{III}, θ^{III} and D

Input: $\theta^{II'}$ of s_i in local coordinate system

Constants : C, c_i, r_i, P, Q, h, and d_i

Constraint Equations: $t_i d_i = Q$

$s_i t_i = P$

$s_i t_i = L$

Loop Equation: $Cc_i + Rr_i + Ss_i + Tt_i + Dd_i = 0$

The input angle $\theta^{II'}$ of s_i is related with R by

$$R = \frac{\theta^{II'}}{2\pi} h \tag{D-1}$$

where h is the pitch of the screw, so R can be readily found.

Choosing the gorund coordinate system with d_3 to coincide with X_3, i.e., $d_1 = d_2 = 0$, and $d_3 = 1$.

From the constraint equation we have

$$t_i d_i = d_3 = \cos \phi^{III} = Q$$

or

$$\phi^{III} = \text{arc cos } (Q) = \text{arc sin} \left[(1 - Q^2)^{\frac{1}{2}}\right] \tag{D-2}$$

Multiplying the loop equation by $\epsilon_{ijk} d_j s_k$ and let $K_i = Cc_i + Rr_i$ then

$$\epsilon_{ijk} d_j s_k (K_i + Tt_i) = 0 \tag{D-3}$$

Since $d_1 = d_2 = 0$, and $d_3 = 1$

Equation (D-3) becomes

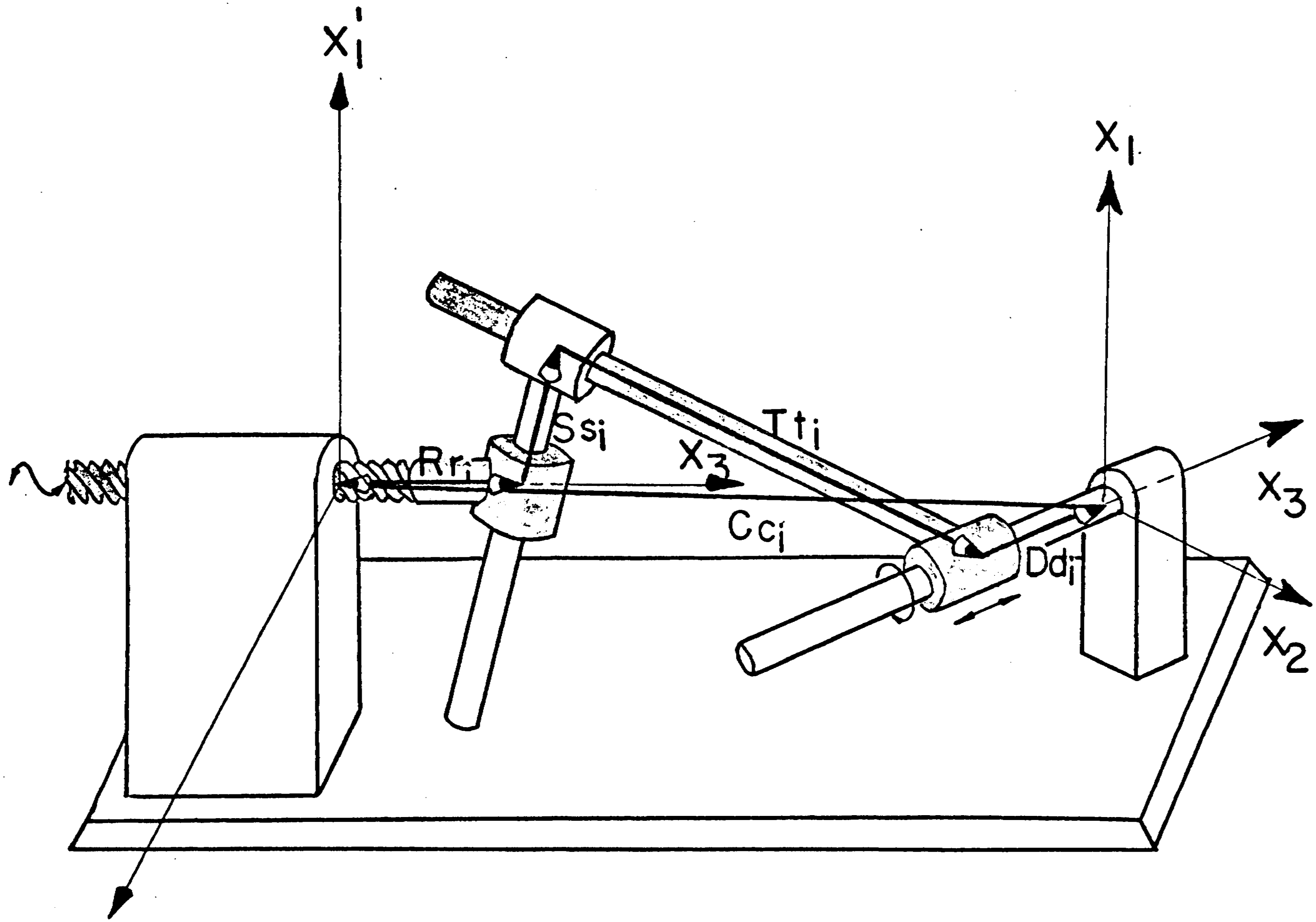

Figure 13. H-C-C-C spatial mechanism

$$\epsilon_{i3k}s_k(K_i + Tt_i) = 0 \tag{D-4}$$

From the definition of ϵ_{ijk} we know equation (D-4) can be expanded as

$$\epsilon_{132}s_2(K_1 + Tt_1) + \epsilon_{231}s_1(K_2 + Tt_2) = 0$$

or

$$s_1\left[K_2 + T(1 - Q^2)^{\frac{1}{2}} \sin \theta^{III}\right] = s_2\left[K_1 + T(1 - Q^2)^{\frac{1}{2}} \cos \theta^{III}\right]$$

Solving for T, we have

$$T = \frac{s_2K_1 - s_1K_2}{(s_1 \sin \theta^{III} - s_2 \cos \theta^{III})(1 - Q^2)^{\frac{1}{2}}} \tag{D-5}$$

Similarly from the constraint $s_it_i = P$ we get

$$s_1 \cos \theta^{III} + s_2 \sin \theta^{III} + \frac{s_3Q - P}{(1 - Q^2)^{\frac{1}{2}}} = 0 \tag{D-6}$$

Solving for θ^{III}

$$\theta^{III} = \text{arc cos}\left\{\frac{s_1(s_3Q - P) \pm s_2\left[(s_1^2+s_2^2)(1-Q^2) - (s_3Q-P)^2\right]}{(s_1^2+s_2^2)(1-Q^2)^{\frac{1}{2}}}\right\} \tag{D-7}$$

The loop equation can be expanded into three simultaneous equations as

$$K_1 + Ss_1 + T \cos \theta^{III} \sin \phi^{III} = 0 \tag{D-8}$$

$$K_2 + Ss_2 + T \sin \theta^{III} \sin \phi^{III} = 0 \tag{D-9}$$

$$K_3 + Ss_3 + T \cos \phi^{III} + D = 0 \tag{D-10}$$

Now substituting T from equation (D-5) into equation (D-8), (D-9), an identical result is obtained for S

$$S = -\frac{K_1 + Tt_1}{s_1} = -\frac{K_2 + Tt_2}{s_2} \tag{D-11}$$

And equation (D-10) gives,

$$D = -\left[K_3 - \frac{K_1 + Tt_1}{s_1} s_3 + \frac{(s_2K_1 - s_1K_2)Q}{(s_1 \sin \theta^{III} - s_2 \cos \theta^{III})(1-Q^2)^{\frac{1}{2}}}\right] \qquad \text{(D-12)}$$

while

$$t_1 = \cos \theta^{III} \sin \phi^{III}, \; t_2 = \sin \theta^{III} \sin \phi^{III}, \text{ and } t_3 = \cos \phi^{III}$$

APPENDIX E

P-C-S-C SPATIAL MECHANISM

Unknowns: S^o, ϕ^{II}, θ^{II}, and T

Input: R

Constants: C, c_i, r_i, s_i^o, S, and t_i

Constraint Equation: $s_i t_i = Q$

Loop Equation: $Cc_i + Rr_i + S^o s_i^o + Ss_i + Tt_i = 0$

Let $K_i = Cc_i + Rr_i$ and the ground coordinate system be assigned that gives $t_3 = 1$, i.e., $\phi^{II} = 0$, therefore

$t_1 = t_2 = 0$

and loop equation in this case becomes

$$K_i + S^o s_i^o + Ss_i + Tt_i = 0 \tag{E-1}$$

From the constraint equation we have

$$s_3 = \cos \phi^{II} = Q$$

or

$$\phi^{II} = \text{arc cos } (Q) = \text{arc sin} \left[(1 - Q^2)^{\frac{1}{2}}\right] \tag{E-2}$$

Multiplying equation (E-1) by t_i and through equation (E-2) we get

$$K_i t_i + S^o s_i^o t_i + QS + T = 0$$

or

$$T = -(K_3 + S^o s_3^o + QS) \tag{E-3}$$

Substituting this into equation (E-1) and expanding

$$K_1 + S^o s_1^o + S(1 - Q^2)^{\frac{1}{2}} \cos \theta^{II} = 0 \tag{E-4}$$

$$K_2 + S^o s_2^o + S(1 - Q^2)^{\frac{1}{2}} \sin \theta^{II} = 0 \tag{E-5}$$

$$K_3 + S^o s_3^o + QS - K_3 - S^o s_3^o - QS = 0 \tag{E-6}$$

Solving for θ^{II} and S^o from equations (E-4) and (E-5),

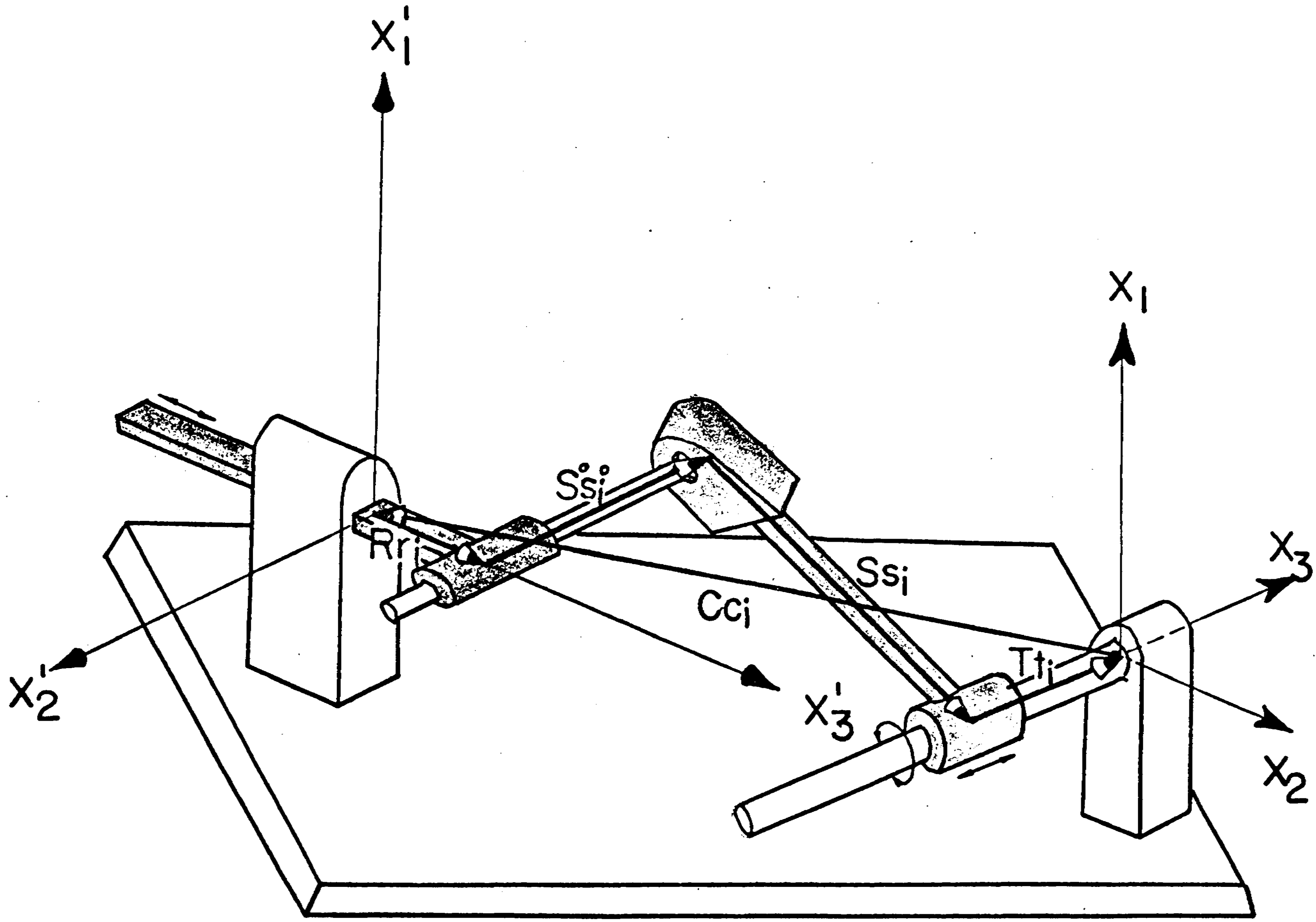

Figure 14. P-C-S-C spatial mechanism

we have

$$\theta^{II} = \text{arc tan}\left(\frac{K_2 + S^o s_2^o}{K_1 + S^o s_1^o}\right) \tag{E-7}$$

$$S^o = \frac{-(K_1 s_1^o + K_2 s_2^o) + \left[(s_1^{o2} + s_2^{o2})S^2(1-Q^2) - s_1^{o2}K_2^2 - s_2^{o2}K_1^2\right]^{\frac{1}{2}}}{s_1^{o2} + s_2^{o2}} \tag{E-8}$$

The minus sign before the square root in equation (E-8) is left out to insure a rational solution, and the following condition must also be met, that is

$$(s_1^{o2} + s_2^{o2})S^2(1 - Q) - s_1^{o2}K_2^2 - s_2^{o2}K_1^2 \geq 0$$

or

$$S^2 \geq \frac{s_1^{o2}K_2^2 - s_2^{o2}K_1^2}{(s_1^{o2} + s_2^{o2})(1 - Q)} \tag{E-9}$$

Knowing S^o from equation (E-8), the values of T, and θ^{II} can be readily gotten by substituting it into equations (E-3) and (E-7).

APPENDIX F

R-C-S-C-R SPATIAL MECHANISM

Unknowns: S, ϕ^{II}, θ^{II}, T, ϕ^{III}, and θ^{III}

Input: r_i

Constants: C, c_i, R

Constraints: s_j is a vector with the given polar angle ϕ^{II^o} and the relation between its azimuthal angle θ^{II^o} and the azimuthal angle θ of r_i, i.e., $\theta^{II^o} = \theta + \alpha$, while s_j is the unit vector of second link Ss_i referring to the input coordinate system. Where α is a given angle.

Loop Equation: $Cc_i + Rr_i + Ss_i - Tt_i = 0$

Similarly we assume $K_i = Cc_i + Rr_i$

From the constraint, s_j can be transformed to the ground to the ground coordinate frame through the transformation matrix (15), that is

$$s_i = A_{ij}s_j$$

or

$$s_1 = A_{11}\sin \phi^{II^o}\cos \theta^{II^o} + A_{12}\sin \phi^{II^o}\sin \theta^{II^o} + A_{13}\cos \phi^{II^o} \tag{F-1}$$

$$s_2 = A_{21}\sin \phi^{II^o}\cos \theta^{II^o} + A_{22}\sin \phi^{II^o}\sin \theta^{II^o} + A_{23}\cos \phi^{II^o} \tag{F-2}$$

$$s_3 = A_{31}\sin \phi^{II^o}\cos \theta^{II^o} + A_{32}\sin \phi^{II^o}\sin \theta^{II^o} + A_{33}\cos \phi^{II^o} \tag{F-3}$$

Thus

$$\phi^{II} = \arccos (s_3) \tag{F-4}$$

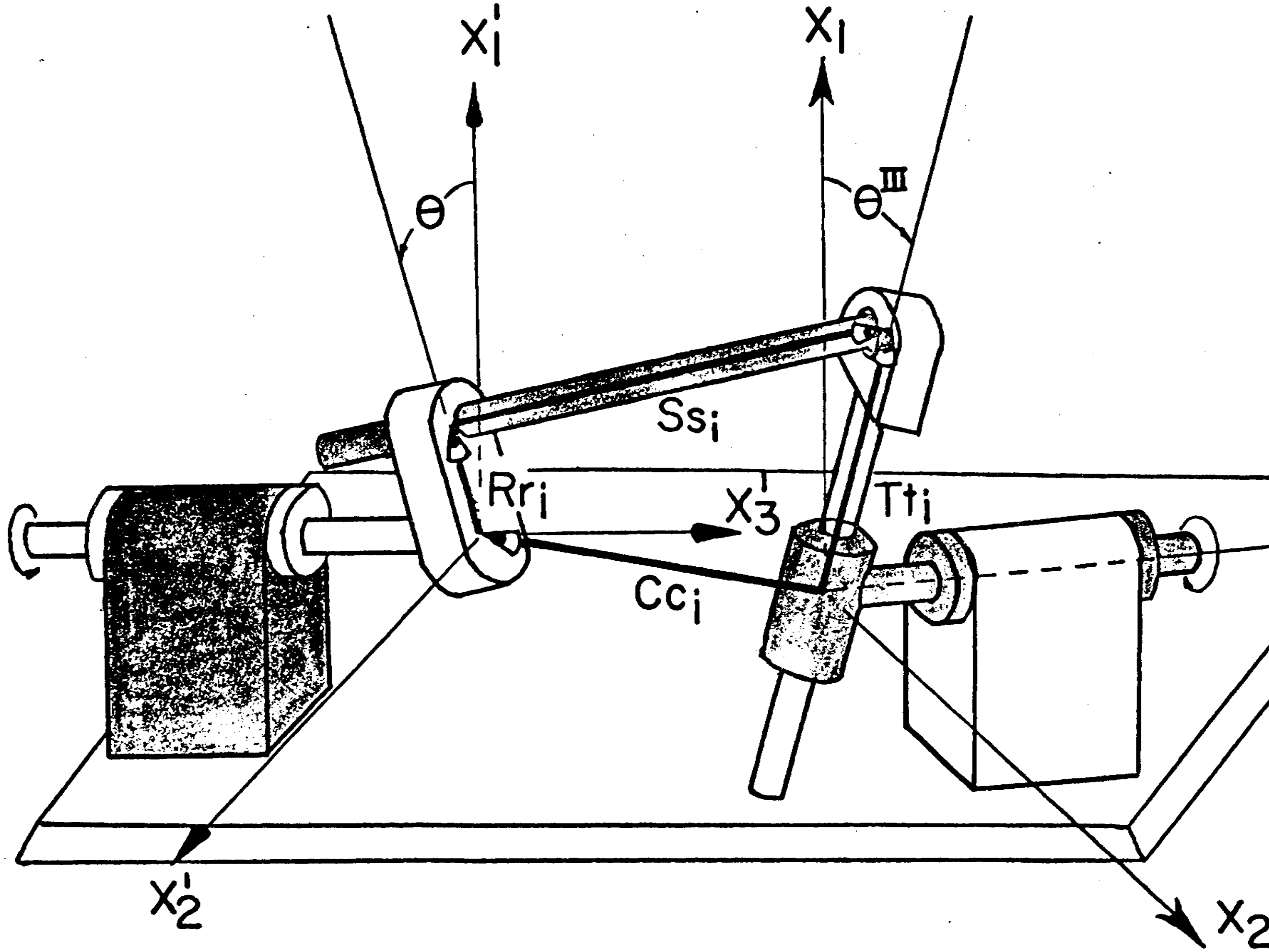

Figure 15. R-C-S-C-R spatial mechanism

$$\theta^{II} = \arc \tan \left(\frac{s_2}{s_1}\right) \quad \text{(F-5)}$$

If the third link rotates about one of the axes of the ground coordinate frame, say x_3, and normal to x_3 then

$\phi^{II} = 90^\circ$

Thus loop equation becomes

$$K_1 + Ss_1 - T \cos \theta^{III} = 0 \quad \text{(F-6)}$$

$$K_2 + Ss_2 - T \sin \theta^{III} = 0 \quad \text{(F-7)}$$

$$K_3 + Ss_3 = 0 \quad \text{(F-8)}$$

Solving for S, θ , and T, we have

$$S = -\frac{K_3}{s_3} \quad \text{(F-9)}$$

$$\theta^{III} = \arc \tan \left(\frac{K_2 s_3 - K_3 s_2}{K_1 s_3 - K_3 s_1}\right) \quad \text{(F-10)}$$

and

$$T = \frac{1}{s_3}\left[(K_1 s_3 - K_3 s_1)^2 + (K_2 s_3 - K_3 s_2)^2\right]^{\frac{1}{2}} \quad \text{(F-11)}$$

APPENDIX G

P-S-P-C SPATIAL MECHANISM

Unknowns: S, ϕ^{II}, θ^{II}, and T

Input: R

Constants: C, c_i, r_i, t_i, and P

Constraint Equation: $s_i t_i = P$

Loop Equation: $Cc_i + Rr_i + Ss_i + Tt_i = 0$

Multiplying the loop equation by t_i, we have

$$Cc_i t_i + Rr_i t_i + PS + T = 0$$

or

$$T = -(Cc_i t_i + Rr_i t_i + PS) \tag{G-1}$$

If $K_i = Cc_i + Rr_i$, the loop equation then becomes

$$K_1 + S \sin \phi^{II} \cos \theta^{II} - (B + PS)t_1 = 0 \tag{G-2}$$

$$K_2 + S \sin \phi^{II} \sin \theta^{II} - (B + PS)t_2 = 0 \tag{G-3}$$

$$K_3 + S \cos \phi^{II} - (B + PS)t_3 = 0 \tag{G-4}$$

where $B = K_i t_i$

Choosing the ground coordinate frame such that $t_3 = 1$, i.e., $\phi^{III} = 0$, therefore

$$t_1 = t_2 = 0$$

The constraint equation gives

$$s_3 = \cos \phi^{II} = P$$

or

$$\phi^{I} = \text{arc cos } (P) = \text{arc sin } \left[(1 - P^2)^{\frac{1}{2}}\right] \tag{G-5}$$

And the equation (G-2) and (G-3) are simplified to be

$$K_1 + S(1 - P^2)^{\frac{1}{2}} \cos \theta^{II} = 0 \tag{G-6}$$

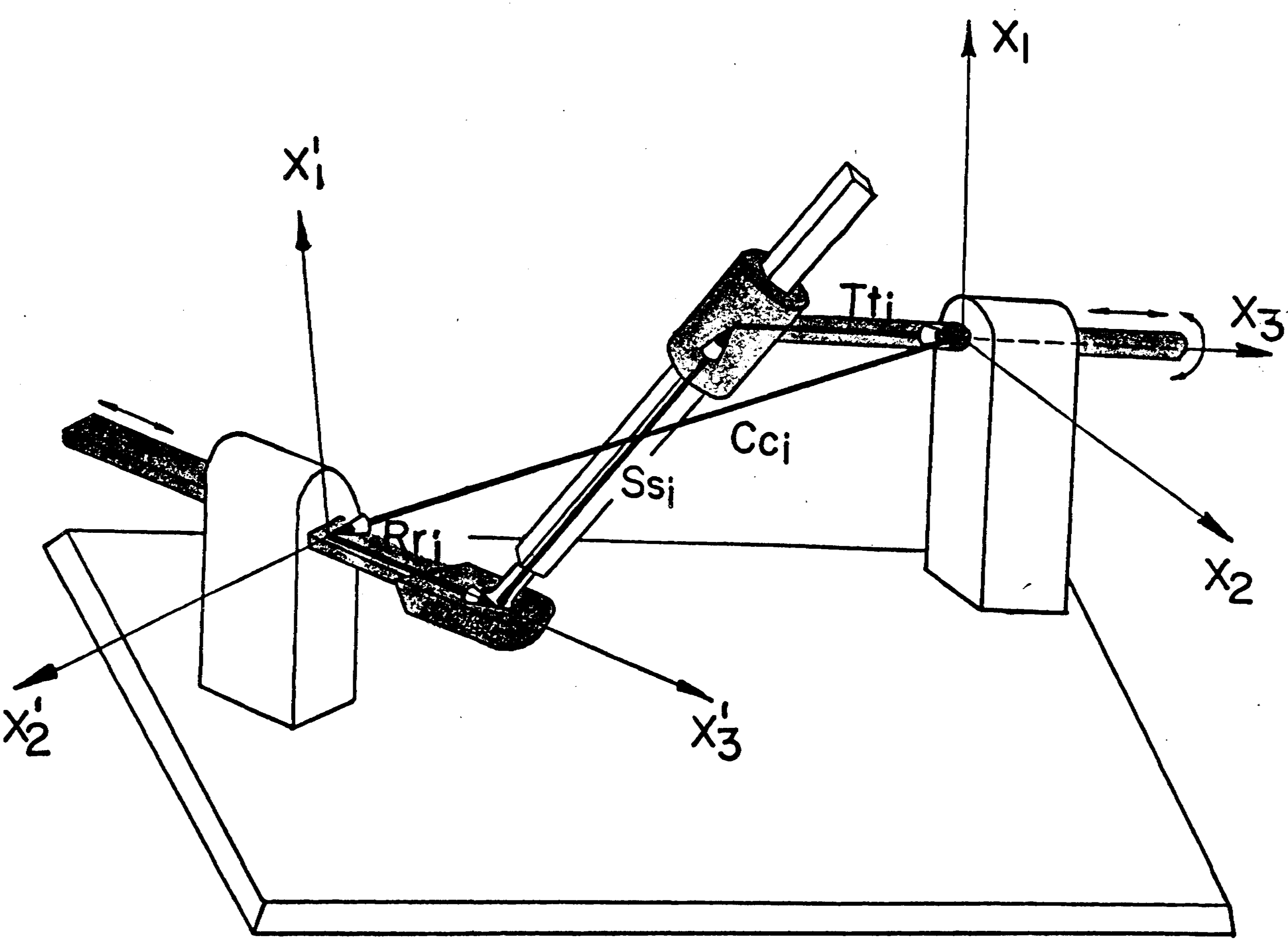

Figure 16. P-S-P-C spatial mechanism

$$K_2 + S(1 - P^2)^{\frac{1}{2}} \sin \theta^{\mathrm{II}} = 0 \tag{G-7}$$

Solving for θ^{II} and S in equation (G-6) and (G-7), we have

$$\theta^{\mathrm{II}} = \arc\tan\,(K_2/K_1) \tag{G-8}$$

$$S = \left[(K_1^2 + K_2^2)/(1 - P^2)\right]^{\frac{1}{2}} \tag{G-9}$$

Thus

$$T = -B - P\left[(K_1^2 + K_2^2)/(1 - P^2)\right]^{\frac{1}{2}} \tag{G-10}$$

APPENDIX H

R-R-C-C-R SPATIAL MECHANISM

Unknowns: S, T, θ^{III} and ϕ^{III}

Input: r_i

Constants: C, c_i, R, S^o, s_i^o, and s_i

Constraints: s_i is always parallel to the axis of rotation of the output link

$$s_i t_i = 0 \tag{H-1}$$

Loop Equation: $Cc_i + Rr_i + S^o s_i^o + Ss_i - Tt_i = 0$

Let $K_i = Cc_i + Rr_i + S^o s_i^o$, then the loop equation becomes

$$K_i + Ss_i - Tt_i = 0 \tag{H-2}$$

In this case, the ground coordinate frame is chosen such that $s_3 = 1$, i.e., $\phi^{II} = 0$, thus

$$s_1 = s_2 = 0$$

Through the constraint equation (H-1), we have

$$t_3 = \cos \phi^{III} = 0, \text{ i.e., } \phi^{III} = 90^o \tag{H-3}$$

Therefore the equation (H-2) can be expanded as

$$K_1 - T \cos \theta^{III} = 0 \tag{H-4}$$

$$K_2 - T \sin \theta^{III} = 0 \tag{H-5}$$

$$K_3 + S = 0 \tag{H-6}$$

From these three equations, we can solve for θ^{III}, T, and S, that is

$$\theta^{III} = \text{arc tan } (K_2/K_1) \tag{H-7}$$

$$T = (K_1^2 + K_2^2)^{\frac{1}{2}} \tag{H-8}$$

$$S = -K_3 \tag{H-9}$$

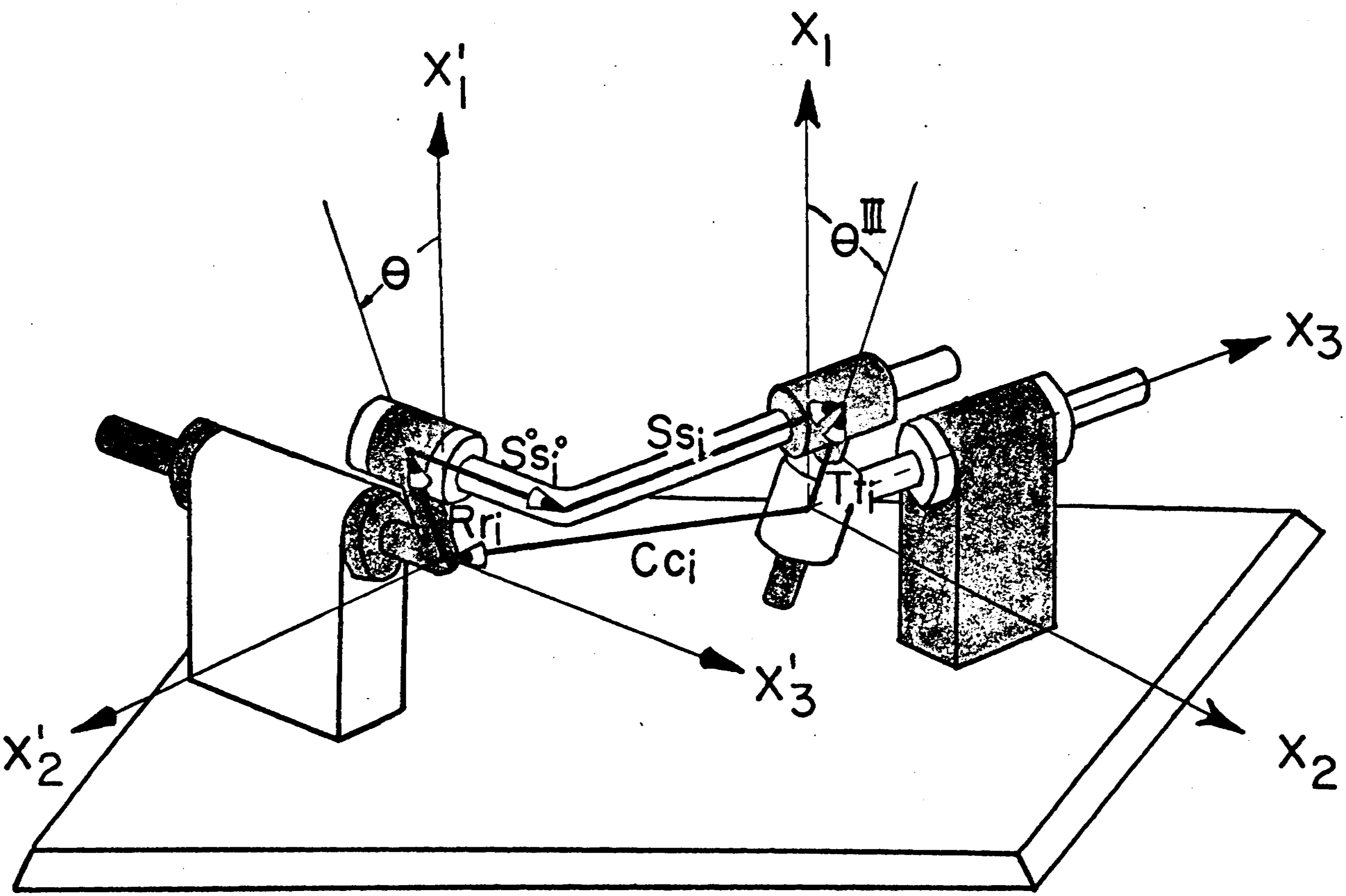

Figure 17. R-R-C-C-R spatial mechanism

The second-order kinematic properties can be readily gotten by differentiating the equations (H-7), (H-8), and (H-9) as

$$\dot{\theta}^{III} = (K_1\dot{K}_2 - K_2\dot{K}_1)/(K_1^2 + K_2^2) \tag{H-10}$$

$$\dot{T} = (K_1\dot{K}_1 + K_2\dot{K}_2)/(K_1^2 + K_2^2)^{\frac{1}{2}} \tag{H-11}$$

$$\dot{S} = -\dot{K}_3 \tag{H-12}$$

and

$$\ddot{\theta}^{III} = \frac{K_1\ddot{K}_2 - K_2\ddot{K}_1}{K_1^2 + K_2^2} - \frac{2(K_1\dot{K}_2 - K_2\dot{K}_1)(K_1\dot{K}_1 + K_2\dot{K}_2)}{(K_1^2 + K_2^2)^2} \tag{H-13}$$

$$\ddot{T} = \frac{\dot{K}_1^2 + \dot{K}_2^2 + \ddot{K}_1K_1 + \ddot{K}_2K_2}{(K_1^2 + K_2^2)^{\frac{1}{2}}} - \frac{(K_1\dot{K}_1 + K_2\dot{K}_2)^2}{(K_1^2 + K_2^2)^{\frac{3}{2}}} \tag{H-14}$$

$$\ddot{S} = -\ddot{K}_3 \tag{H-15}$$

APPENDIX I

COMPUTER PROGRAM

Fortran Variable Names

The following is a list of the important variables of the program and what they represent.

1) AI's — components of the direction vector of the driving link with respect to the input coordinate system

2) B's — matrix of transformation for the input coordinate system

3) CI's — components of the direction vector of the ground link

4) CITA — input angle, θ, or azimuthal angle of the driving link with respect to the input coordinate system

5) CITAO — azimuthal angle of the unit vector of the input shaft, θ'

6) CITA3 — azimuthal angle of the direction vector of the driven link, θ^{III}

7) CITA4 — azimuthal angle of the ground link, θ^{IV}

8) DCITA — angular velocity of the driving link, $\dot{\theta}$

9) DCITA3 — output angular velocity

10) DDC3 — output angular acceleration

11) DDCITA — angular acceleration the driving link, $\ddot{\theta}$

12) DDSO — sliding acceleration of the first part of link 2

13) DDT — sliding acceleration of link 3

14) DSO — sliding velocity of the first part of link 2

15) DT — sliding velocity of link 3

16) FF12 — friction force due to the sliding motion of link 2 in the first cylindrical pair

17) FF34 — friction force due to the sliding motion of link 3 in the second cylindrical pair

18) GI's — components of the direction vector of K_i

19) PHAO — polar angle of the input shaft, ϕ'

20) PHA4 — polar angle of the ground link, ϕ^{IV}

21) R — length of link 1 or the driving link

22) RI's — components of the direction vector of the driving link with respect to the ground coordinate system

23) S — length of the second part of link 2

24) SO — length of the first part of link 2

25) SOI's — components of the direction vector of the first part of link 2

26) T — length of link 3

27) TF12 — couple produced by the first cylindrical pair due to bearing friction

28) TF23 — couple produced by the second revolute pair due to bearing friction

29) TF45 — couple produced by the third revolute pair due to bearing friction

30) TF51 — couple produced by the first revolute pair due to bearing friction

31) THATA — input angle, θ

32) TI51 — input torque acting on shaft of link 1

33) TO45 — output torque from shaft of link 4

```
                 C       MECS490H HSEI KAO-CHIEN  R-C-R-C-R  SPATIAL MECHANISM
 0001                    DIMENSION AI(3),B(3,3),CI(3),GI(181),RI(181),THATA(181),SO(181),T(
                        1181),CITA3(181),SOI(181),SI(3),DCITA3(181),DSO(181),DT(181),DDC3(1
                        281),DDT(181),DGI(3),DDGI(3),DDSO(181),T045(181),TI(3),DTI(3),DDTI(
                        13),DQ(181),DDQ(181)
 0002                    PP=3.14159265/180.
 0003                    PHAO=30.
 0004                    WRITE(3,333) PHAO
 0005                    R=1.
 0006                    C=2.5
 0007                    DCITA=10.
 0008                    DDCITA=0.
 0009                    TI51=1.
 0010                    TF51=0.01
 0011                    FF12=0.03
 0012                    FF34=0.03
 0013                    TF12=0.01
 0014                    TF23=0.01
 0015                    TF45=0.01
 0016                    CITA0=90.*PP
 0017                    PHA0=PHA0*PP
 0018                    CITA4=270.*PP
 0019                    PHA4=150.*PP
 0020                    SNC0=SIN(CITA0)
 0021                    CSC0=COS(CITA0)
 0022                    SNP0=SIN(PHA0)
 0023                    CSP0=COS(PHA0)
 0024                    SNC4=SIN(CITA4)
 0025                    CSC4=COS(CITA4)
 0026                    SNP4=SIN(PHA4)
 0027                    CSP4=COS(PHA4)
 0028                    SOI(1)=SNP0*CSC0
 0029                    SOI(2)=SNP0*SNC0
 0030                    SOI(3)=CSP0
 0031                    B(1,1)=SNC0
 0032                    B(1,2)=-CSC0
 0033                    B(1,3)=0.
 0034                    B(2,1)=CSC0*CSP0
 0035                    B(2,2)=SNC0*CSP0
 0036                    B(2,3)=-SNP0
 0037                    B(3,1)=CSC0*SNP0
 0038                    B(3,2)=SNC0*SNP0
 0039                    B(3,3)=CSP0
 0040                    CI(1)=SNP4*CSC4
 0041                    CI(2)=SNP4*SNC4
 0042                    CI(3)=CSP4
 0043                    H=SOI(3)/SOI(2)
 0044                    S=C*(CI(2)*H-CI(3))
 0045                    WRITE(3,444) S
```

```
      WRITE(3,111)
      DO 3 I=1,181
      THATA(I)=2.*(I-1)
      CITA=(I-1)*2.*PP
      AI(1)=COS(CITA)
      AI(2)=SIN(CITA)
      AI(3)=0.
      SNC=SIN(CITA)
      CSC=COS(CITA)
      DO 2  J=1,3
      SUM=0.
      DO 1  K=1,3
    1 SUM=SUM+AI(K)*B(K,J)
      RI(J)=SUM
    2 GI(J)=C*CI(J)+R*RI(J)
      SO(I)=-(GI(3)+S)/SOI(3)
      G1=GI(1)+SO(I)*SOI(1)
      G2=GI(2)+SO(I)*SOI(2)
      GG=ABS(G2/G1)
      CITA3(I)=ATAN(GG)/PP
      IF(G1) 51,56,54
   51 IF(G2) 53,56,52
   52 CITA3(I)=180.-CITA3(I)
      GO TO 56
   53 CITA3(I)=180.+CITA3(I)
      GO TO 56
   54 IF(G2) 55,56,56
   55 CITA3(I)=360.-CITA3(I)
   56 T(I)=(G1*G1+G2*G2)**0.5
      C3=CITA3(I)*PP
      CC3=COS(C3)
      SC3=SIN(C3)
      TI(1)=CC3
      TI(2)=SC3
      TI(3)=0.
      E=DDCITA/DCITA
      F=R*DCITA**2
      DGI(1)=DCITA*(CSCO*CSPO*CSC-SNCO*SNC)*R
      DGI(2)=DCITA*(CSCO*SNC+SNCO*CSPO*CSC)*R
      DGI(3)=-DCITA*SNPO*CSC*R
      DDGI(1)=E*DGI(1)-F*(SNCO*CSC+CSCO*CSPO*SNC)
      DDGI(2)=E*DGI(2)+F*(CSCO*CSC-SNCO*CSPO*SNC)
      DDGI(3)=E*DGI(3)+F*SNPO*SNC
      DSO(I)=-DGI(3)/SOI(3)
      DG1=DGI(1)+DSO(I)*SOI(1)
      DG2=DGI(2)+DSO(I)*SOI(2)
      DCITA3(I)=(G1*DG2-G2*DG1)/T(I)**2
      DT(I)=(G1*DG1+G2*DG2)/T(I)
      DTI(1)=-SC3*DCITA3(I)
      DTI(2)=CC3*DCITA3(I)
      DTI(3)=0.
```

```
      A1=DT(I)*TI(1)+T(I)*DTI(1)
      A2=DT(I)*TI(2)+T(I)*DTI(2)
      A3=0.
      DQ(I)=SQRT(A1*A1+A2*A2+A3*A3)
      DDSO(I)=-DDGI(3)/SOI(3)
      DDG1=DDGI(1)+DDSO(I)*SOI(1)
      DDG2=DDGI(2)+DDSO(I)*SOI(2)
      DDC3(I)=(G1*DDG2-G2*DDG1)/T(I)**2-2.*DCITA3(I)*DT(I)/T(I)
      DDT(I)=(DG1**2+DG2**2+G1*DDG1+G2*DDG2-DT(I)**2)/T(I)
      DDTI(1)=-CC3*DCITA3(I)**2-SC3*DDC3(I)
      DDTI(2)=-SC3*DCITA3(I)**2+CC3*DDC3(I)
      DDTI(3)=0.
      AA1=DDT(I)*TI(1)+T(I)*DDTI(1)+2.*DT(I)*DTI(1)
      AA2=DDT(I)*TI(2)+T(I)*DDTI(2)+2.*DT(I)*DTI(2)
      AA3=0.
      DDQ(I)=SQRT(AA1*AA1+AA2*AA2+AA3*AA3)
      TO45(I)=((TI51-TF51)*DCITA-FF12*ABS(DSO(I))-FF34*ABS(DT(I)))/DCITA
     53(I)-TF12-TF23-TF45
    3 WRITE(3,222) THATA(I),CITA3(I),DCITA3(I),DDC3(I),TO45(I),DQ(I),DDQ
     5(I)
  111 FORMAT(10X,'INPUT ANGLE',3X,'OUTPUT ANGLE',3X,'OUTPUT ANG. VEL.',4
     3X,'OUTPUT ANG. ACCE.',4X,'OUTPUT TORQUE',4X,'VEL. OF Q',4X,'ACCE.
     6OF Q',////)
  222 FORMAT(12X,F5.1,6(7X,F10.5))
  333 FORMAT(1H1,40X,'PHAO=',F10.5,///)
  444 FORMAT(40X,'S= ',F10.5,////)
      CALL PPLT(THATA,TO45,181)
      CALL PPLT(THATA,DQ,181)
      CALL PPLT(THATA,DDQ,181)
      STOP
      END
```

APPENDIX J

FIGURES

OF SOME VARIABLES OF

R-C-R-C-R SPATIAL MECHANISM

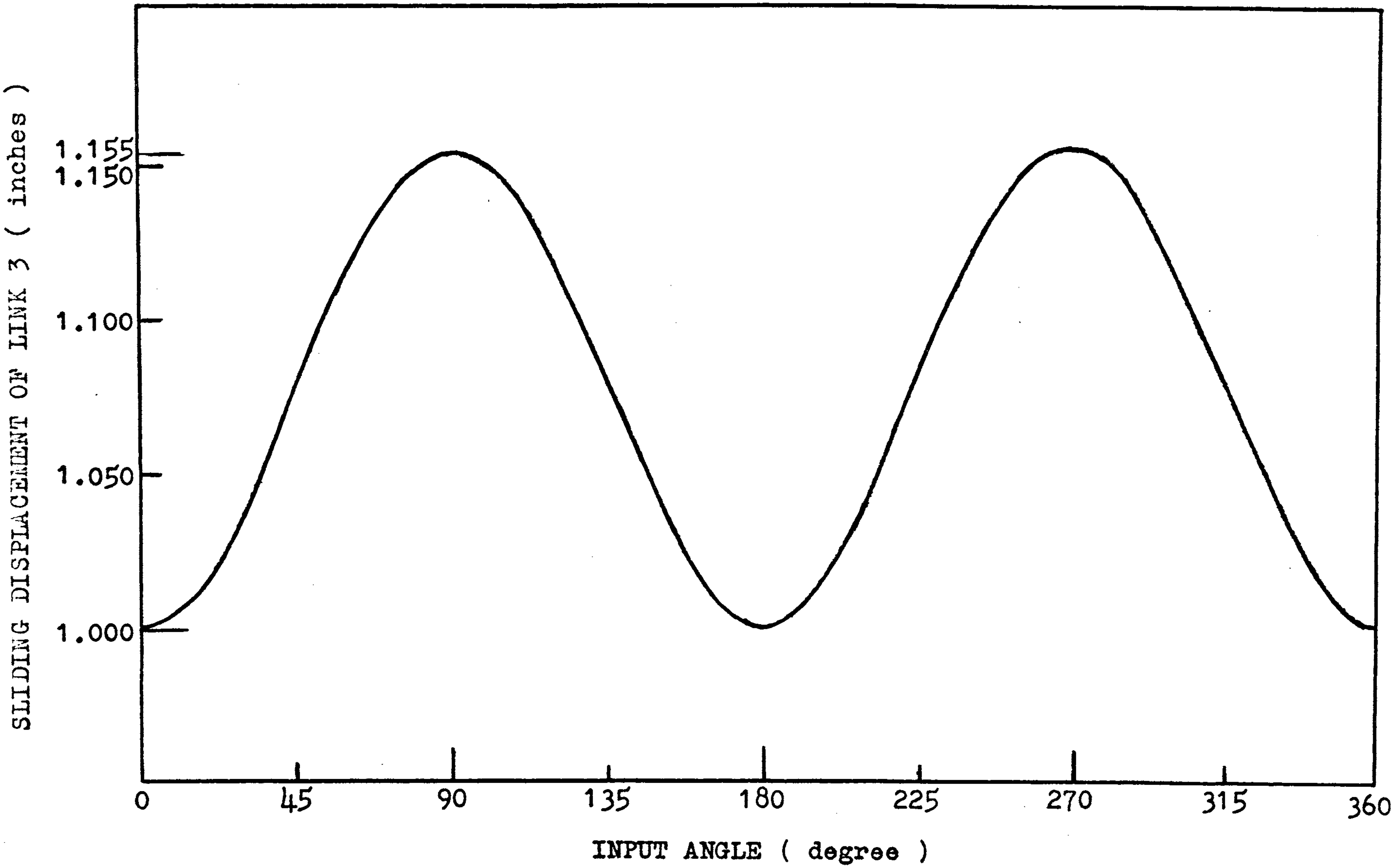

Figure 18. Length of link 3 versus θ

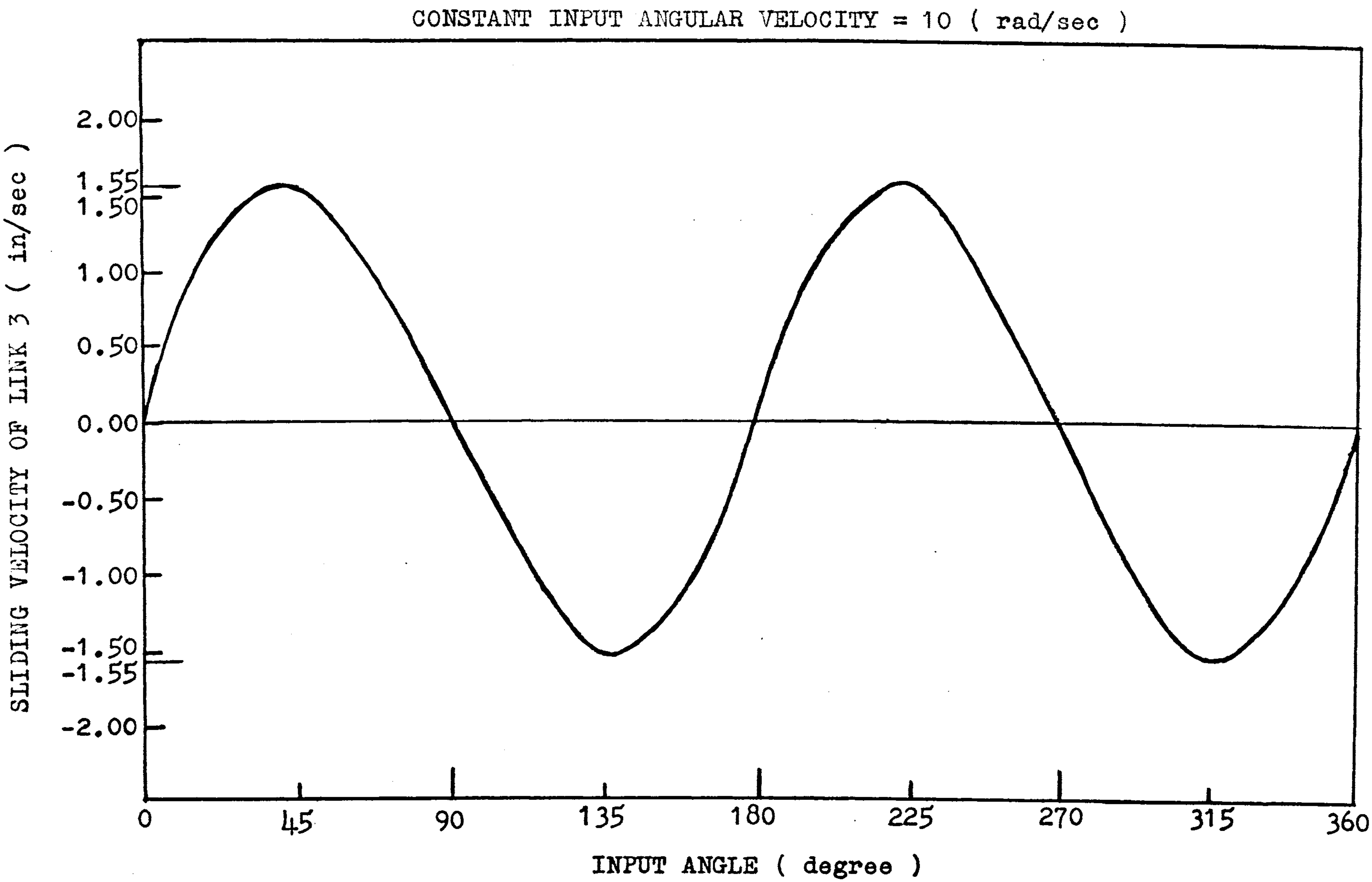

Figure 19. Sliding velecity of link 3 versus θ

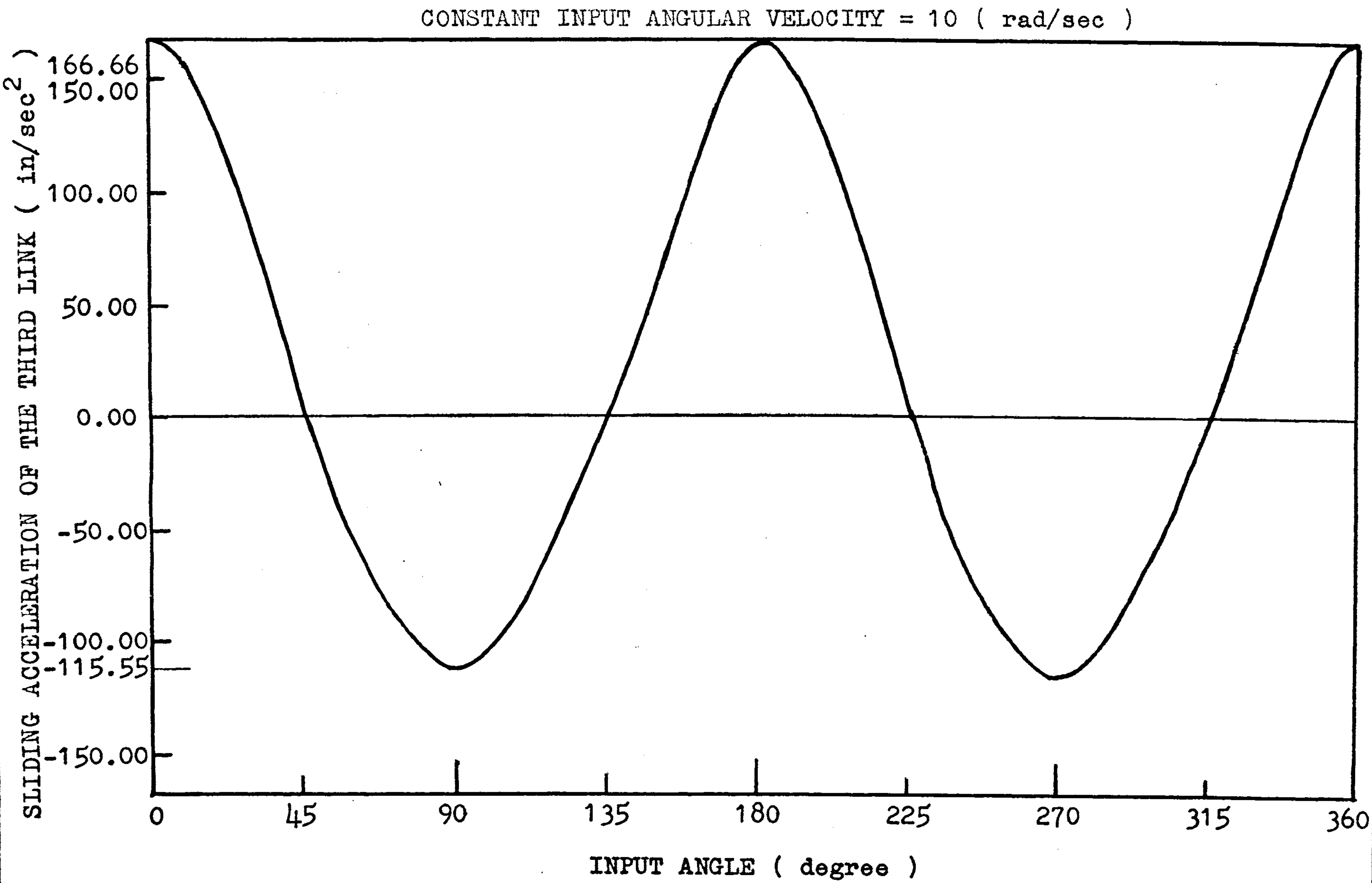

Figure 20. Sliding acceleration of link 3 versus θ

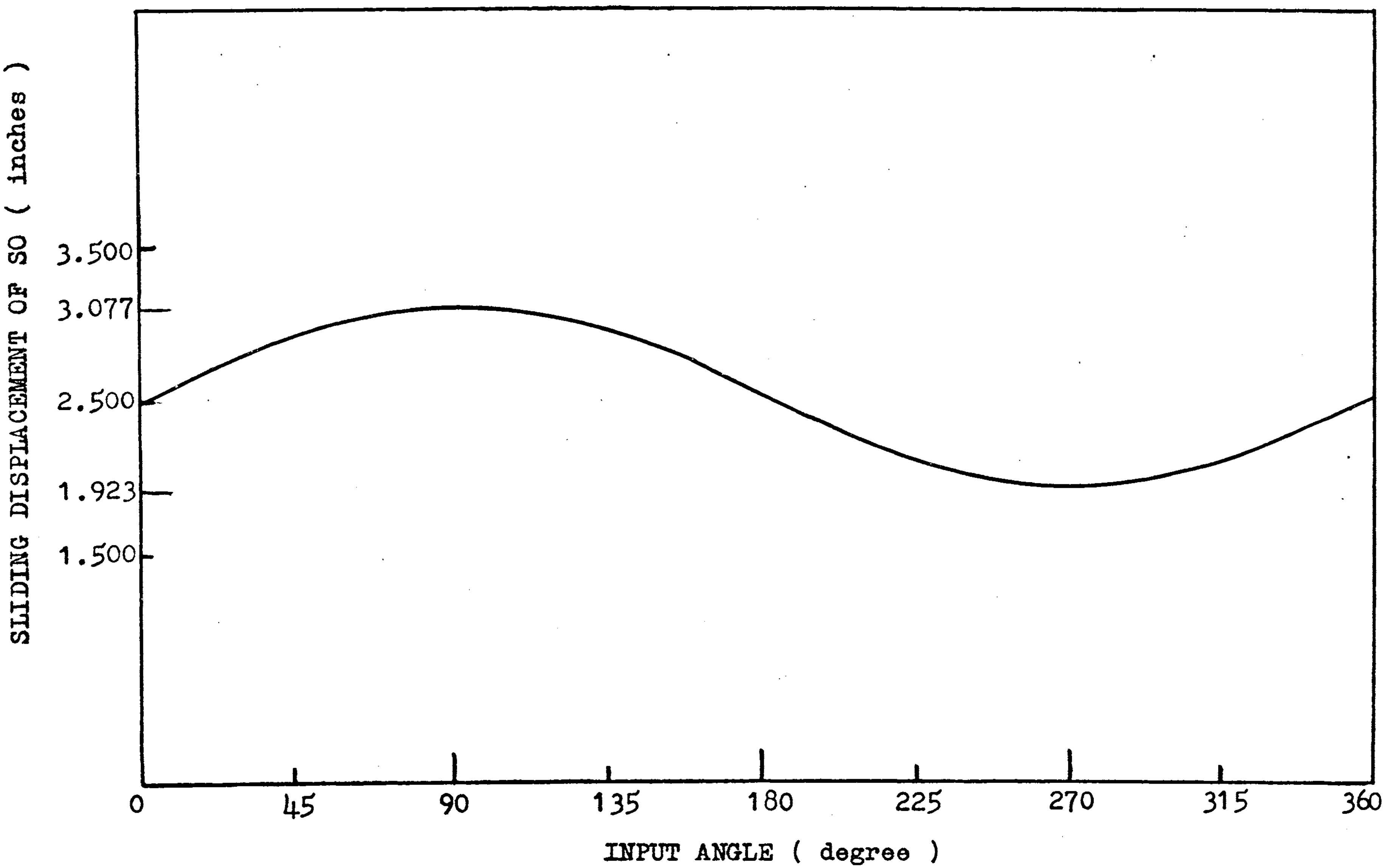

Figure 21. Length of SO versus θ

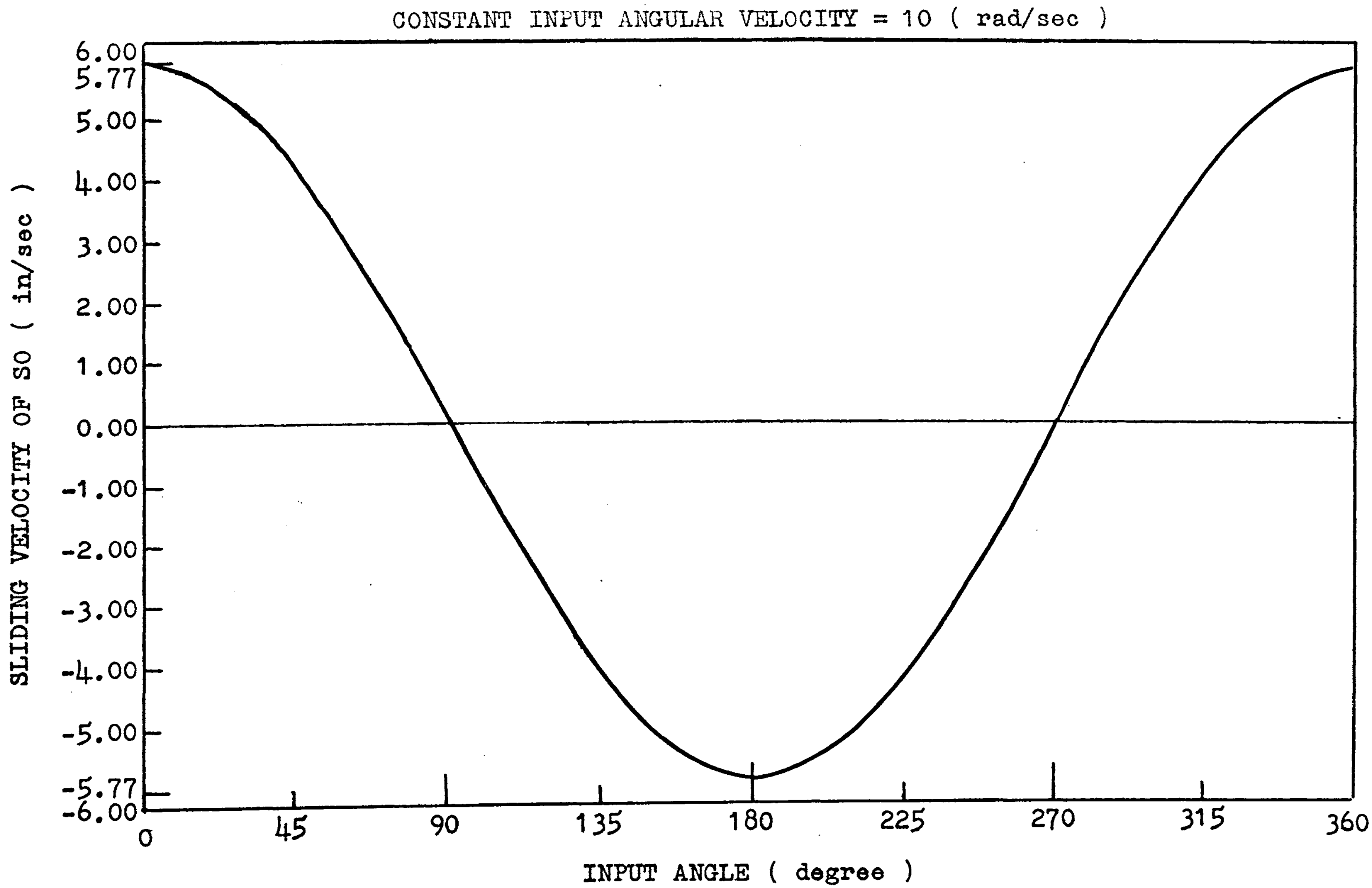

Figure 22. Sliding velocity of SO versus θ

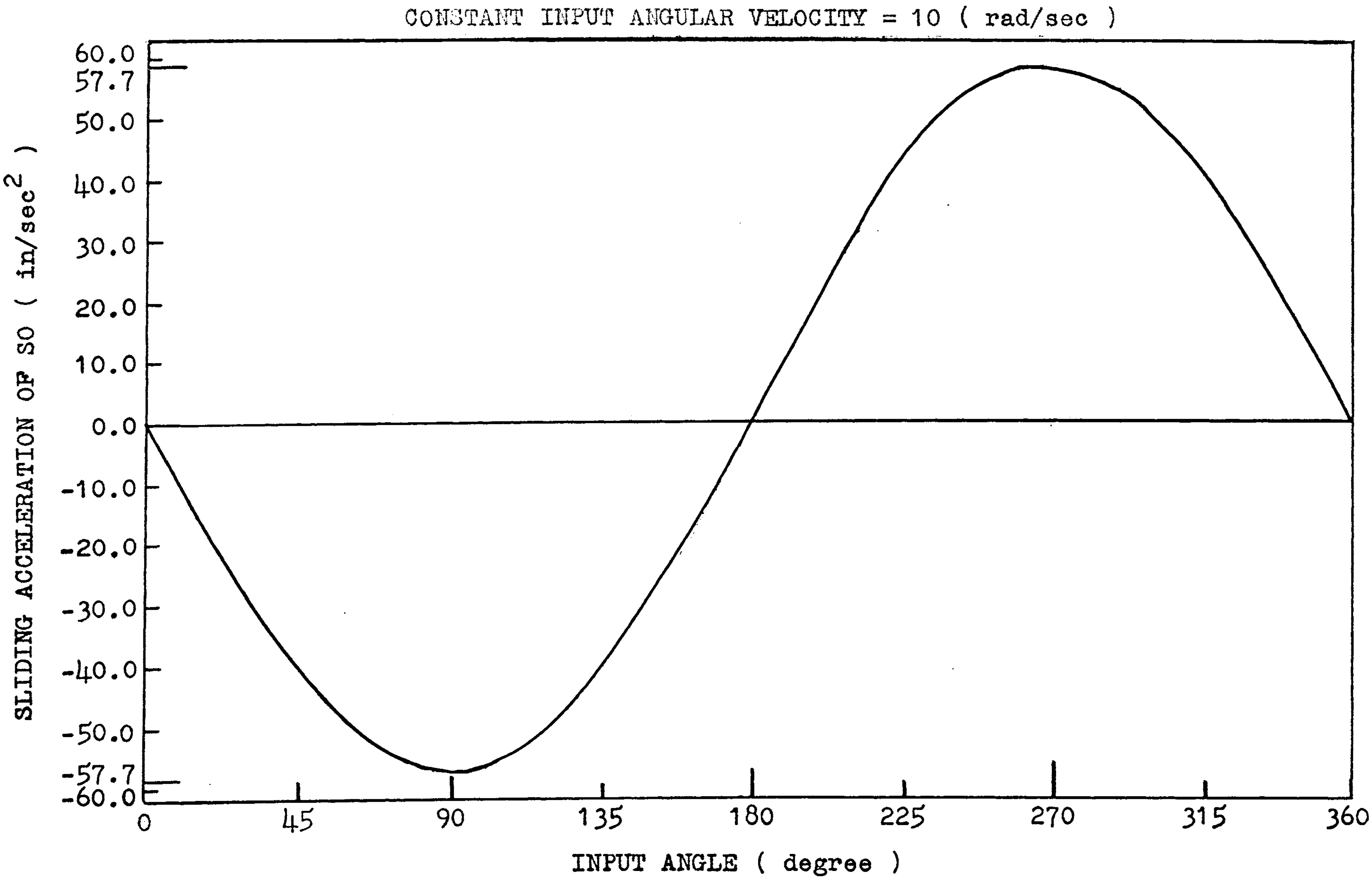

Figure 23. Sliding acceleration of SO versus θ

VITA

Kao-chien Hsei was born in Kweilin, Kwangsi, China on November 7, 1943. After graduating from Taiwan Provincial Ping-Tung High School in 1961, he went to Taiwan Provincial Cheng-Kung University where he received his Bachelor of Science Degree in Mechanical Engineering in June, 1965.

In September 1967 he enrolled in the Graduate School of the University of Missouri at Rolla and began a program leading to the Master of Science Degree in Mechanical Engineering.

He has held the position of methods engineer and manufacturing engineer while employed by Taiwan Electronics Corporation and Philco-Taiwan Corporation, subsidiaries of General Instruments Corporation and Philco-Ford Corporation, U.S.A., and has been a graduate teaching assistant at the University of Missouri at Rolla.

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